# Tracking by Regression

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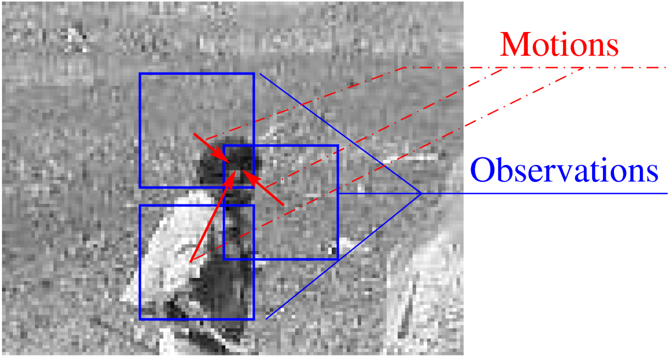
March 31, 2014

# Linear mapping for tracking




$\Phi($$) = (0, 0)^T$ $\Phi($$) = (12, 7)^T$

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# Optimization vs. regression based tracking

KLT:

$$\Delta \mathbf{p}^* = \operatorname*{argmin}_{\Delta \mathbf{p} \in S} \sum_{\mathbf{x}} [I(\mathbf{W}(\mathbf{x}; \mathbf{p} + \Delta \mathbf{p})) - T(\mathbf{x})]^2$$

Optimization in general:

$$\mathbf{p}^* = \operatorname*{argmin}_{\mathbf{p} \in S} f(\mathbf{p}; I, \mathbf{p}_0),$$

Regresssion based tracking:

$$\mathbf{p}^* = \varphi(I, \mathbf{p}_0)$$

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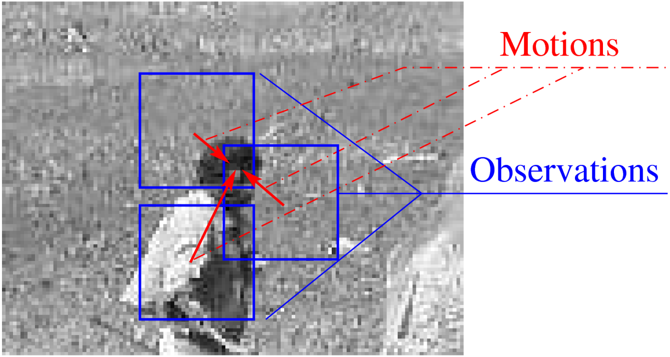
# Regression learned from data - sample image

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# Regression learning

$$\varphi^* = \operatorname*{argmin}_{\varphi} \sum_{\mathbf{p}} \|\varphi\Big(I(\mathbf{p} \circ \mathbf{x})\Big) - \mathbf{p}\|.$$

# Connection to KLT

$$\Delta\mathbf{p} = \mathtt{H}^{-1} \sum_{\mathbf{x}} \left[ \nabla I \, \frac{\partial \mathbf{W}}{\partial \mathbf{p}} \right]^{\top} [T(\mathbf{x}) - I(\mathbf{W}(\mathbf{x};\mathbf{p}))]$$

where:

$$\mathtt{H} = \sum_{\mathbf{x}} \left[ \nabla I \, \frac{\partial \mathbf{W}}{\partial \mathbf{p}} \right]^{\top} \left[ \nabla I \, \frac{\partial \mathbf{W}}{\partial \mathbf{p}} \right]$$

Reformulating *regression*:

$$\mathbf{p} = \varphi\Big(I(\mathbf{x})\Big) = \mathtt{H}\Big(I(\mathbf{x}) - T(\mathbf{x})\Big)$$

How to get $\mathtt{H}$?

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# Collecting training data

Image template: $T = T(\mathbf{x})$
Collected training pairs $(I^i,\ \mathbf{p}^i)$ $(i = 1 \ldots d)$ of observed intensities $I^i$ and corresponding motion parameters $\mathbf{p}^i$, which align the object with current frame.

*Training set* is an ordered pair $(\mathtt{I}, \mathtt{P})$, such that $\mathtt{I} = [\mathbf{I}^1 - \mathbf{T}, \mathbf{I}^2 - \mathbf{T}, \ldots \mathbf{I}^d - \mathbf{T}]$ and $\mathtt{P} = [\mathbf{p}^1, \mathbf{p}^2, \ldots \mathbf{p}^d]$.

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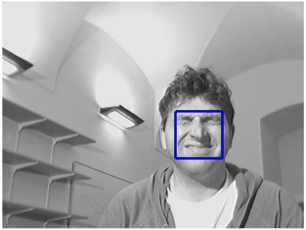
# Learning alignment for one predictor



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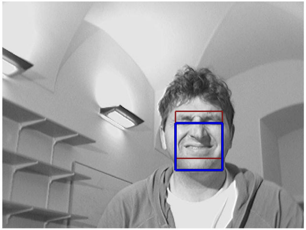

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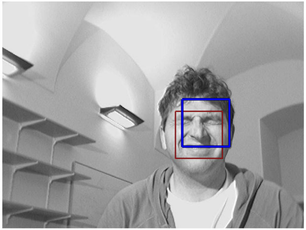

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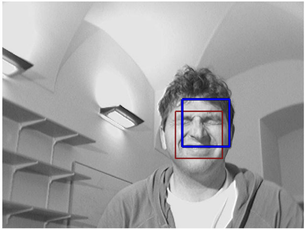

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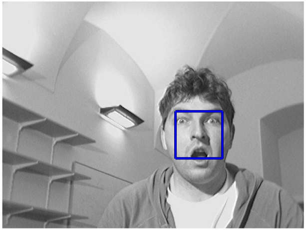

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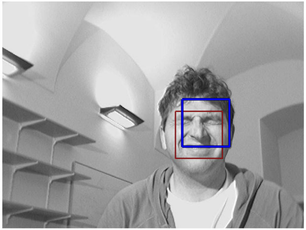

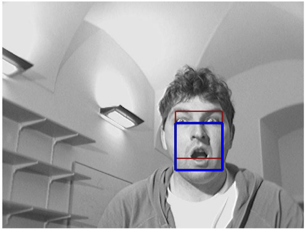

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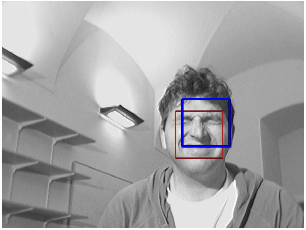

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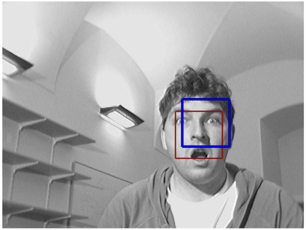

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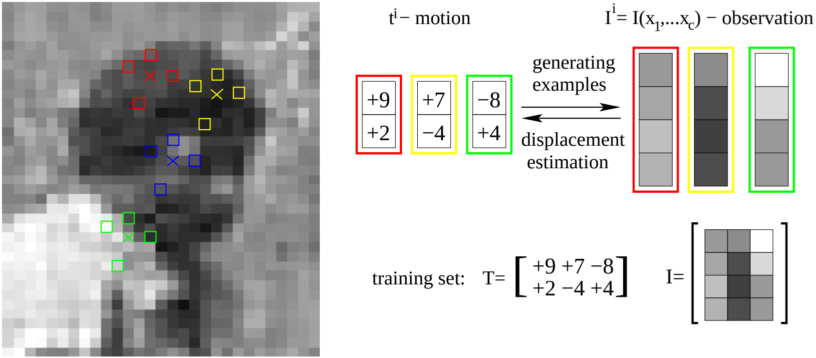
# Generating training examples




Training set: $(\mathtt{I}, \mathtt{P})$
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# Learning $\mathtt{H}$ from data

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\begin{aligned}
\mathtt{H}^* &= \operatorname*{argmin}_{\mathtt{H}} \sum_{i=1}^{d} \|\mathtt{H}(\mathbf{I}^i - \mathbf{T}) - \mathbf{p}^i\|_2^2 = \operatorname*{argmin}_{\mathtt{H}} \|\mathtt{HI} - \mathtt{P}\|_F^2 = \\
&= \operatorname*{argmin}_{\mathtt{H}} \operatorname{trace}(\mathtt{HI} - \mathtt{P})(\mathtt{HI} - \mathtt{P})^\top = \\
&= \operatorname*{argmin}_{\mathtt{H}} \operatorname{trace}(\mathtt{HII}^\top\mathtt{H}^\top - 2\mathtt{HIP}^\top + \mathtt{PP}^\top).
\end{aligned}
$$

Next its derivative is set equal to zero:

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2\mathtt{H}^*\mathtt{II}^\top - 2\mathtt{PI}^\top &= 0 \\
\mathtt{H}^*\mathtt{II}^\top &= \mathtt{PI}^\top \\
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# LS learning - summary

$$\varphi^* = \mathrm{argmin}_{\varphi} \sum_{\mathbf{p}} \|\varphi\Big(I(\mathbf{p} \circ \mathbf{x})\Big) - \mathbf{p}\|^2.$$

Minimizes sum of square errors over all training set. Leads to matrix pseudoinverse computation.

Example for *linear mapping*:

$$\mathtt{H}^* = \underset{\mathtt{H}}{\mathrm{argmin}} \sum_{i=1}^{d} \|\mathtt{H}(\mathbf{I}^i - \mathbf{T}) - \mathbf{t}^i\|_2^2 = \underset{\mathtt{H}}{\mathrm{argmin}} \|\mathtt{HI} - \mathtt{P}\|_F^2$$

after some derivation

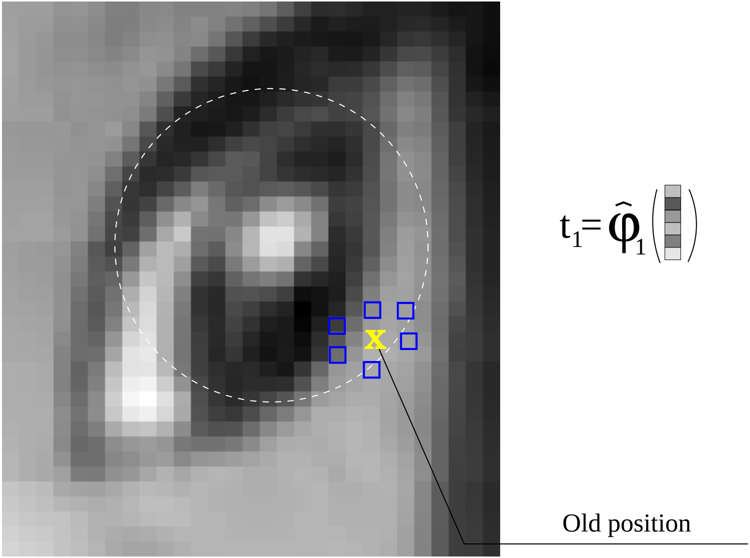
$$\mathtt{H}^* = \mathtt{P}\underbrace{\mathtt{I}^\top(\mathtt{II}^\top)^{-1}}_{\mathtt{I}^+} = \mathtt{PI}^+.$$

# Min-max learning

$$\varphi^* = \mathrm{argmin}_{\varphi} \max_{\mathbf{p}} \left\|\varphi\Big(I(\mathbf{p} \circ \mathbf{x})\Big) - \mathbf{p}\right\|_\infty .$$

Minimizes the *worst case* (the biggest estimation error) in the training set. Leads to linear programming.

# Tracking of a single point by a *sequence* of predictors



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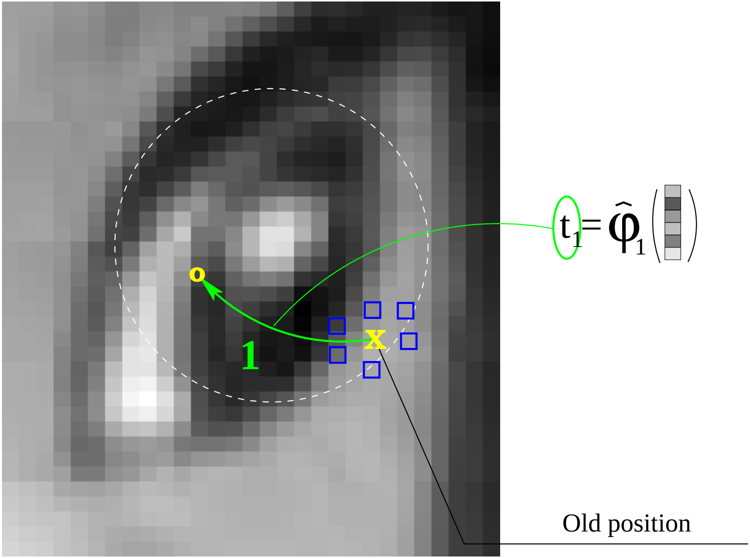

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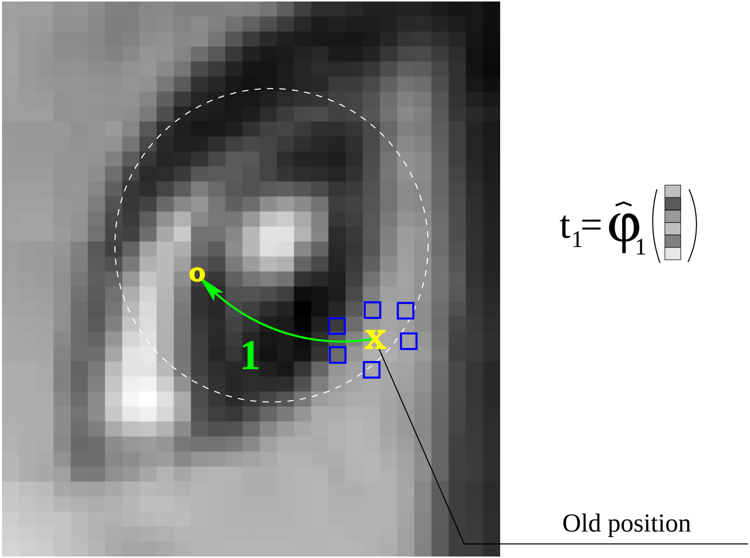


$$t_1 = \hat{\varphi}_1\left(\,\cdot\,\right)$$

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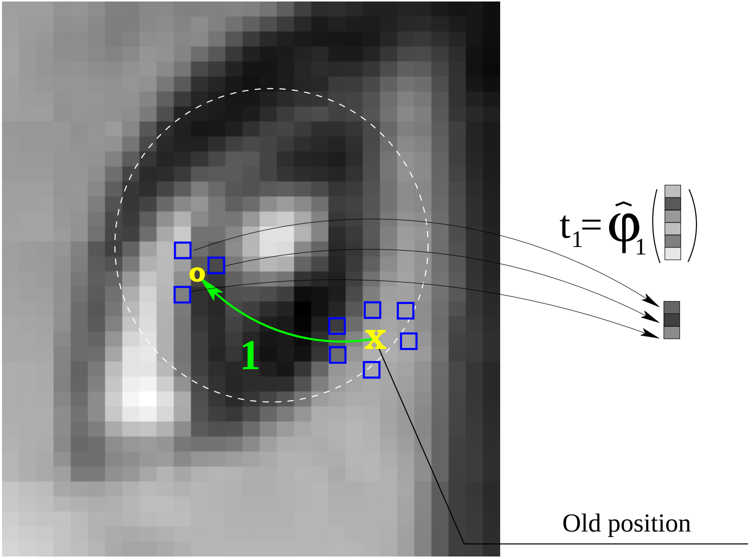

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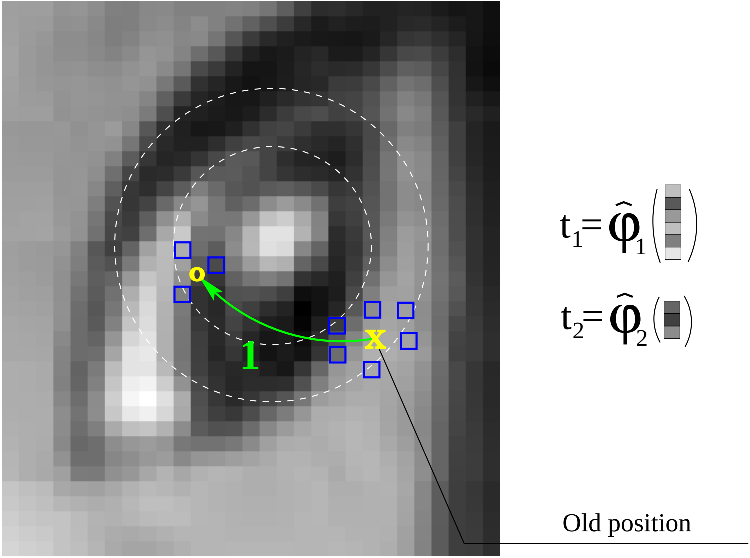


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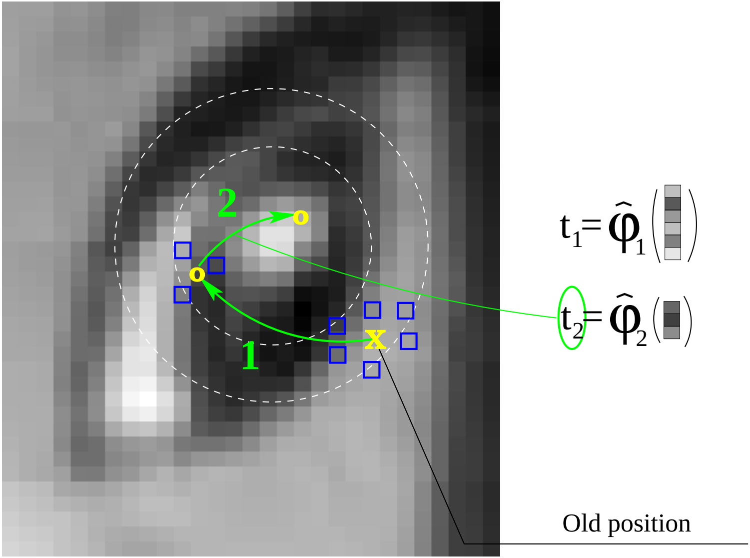

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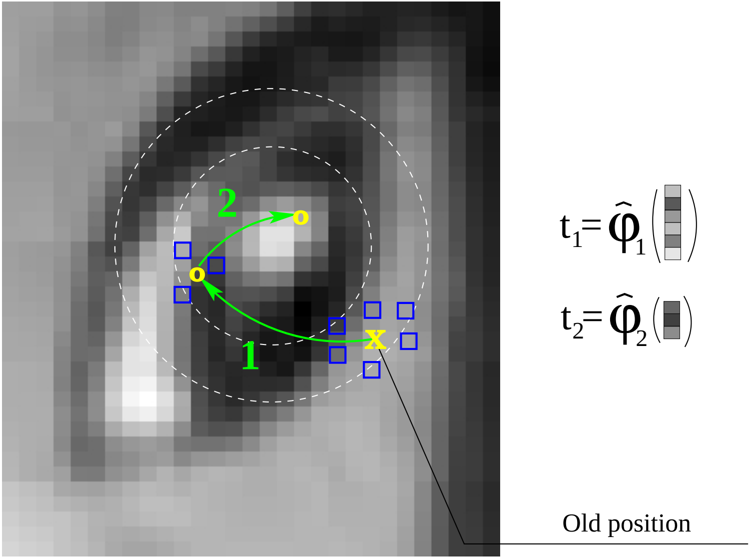

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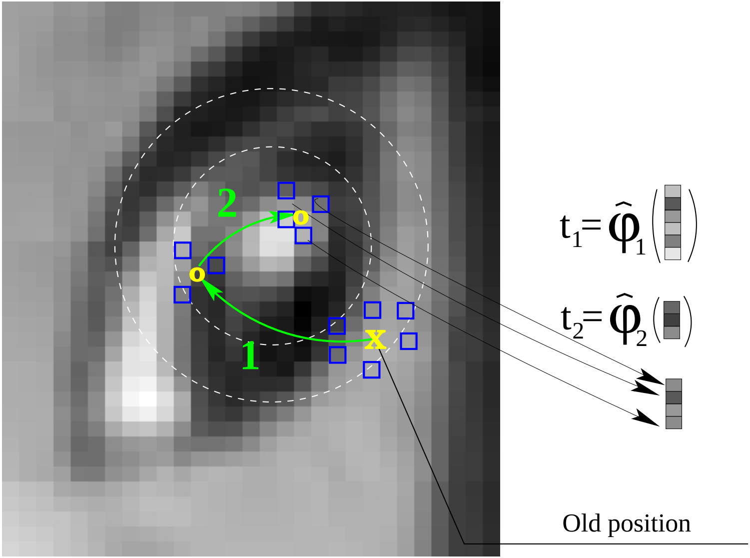

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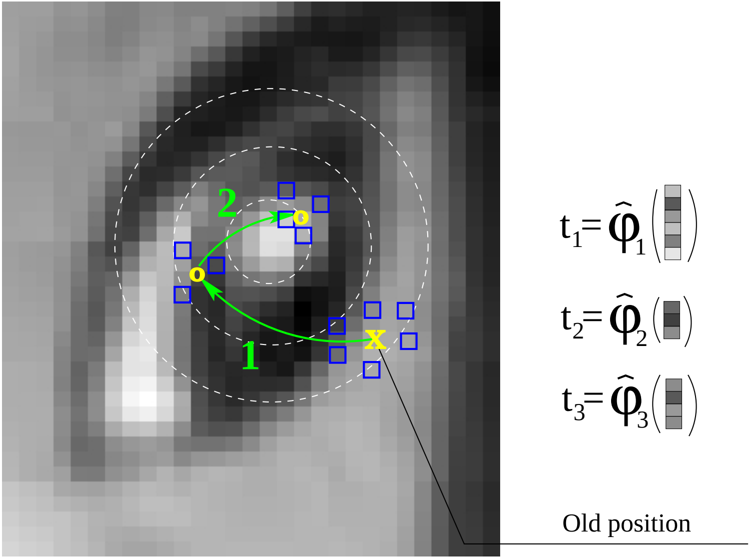

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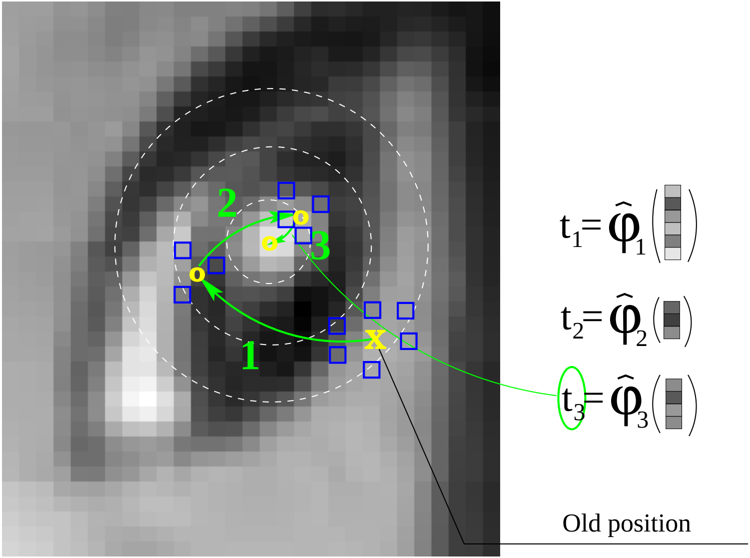

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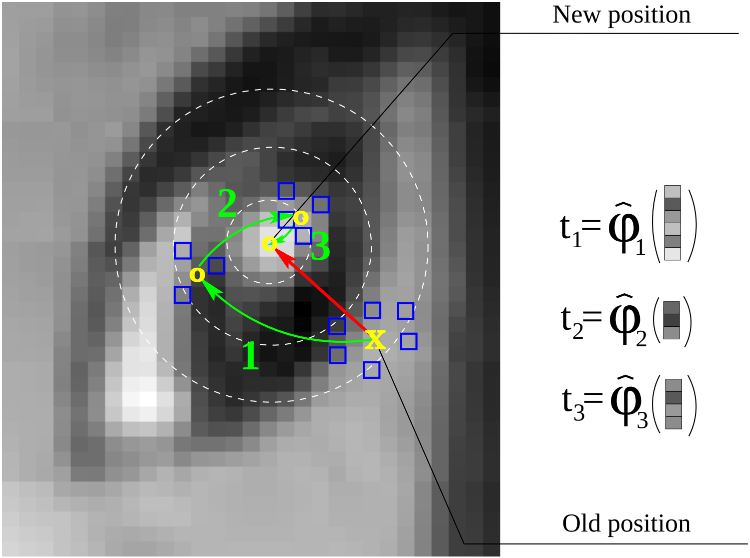

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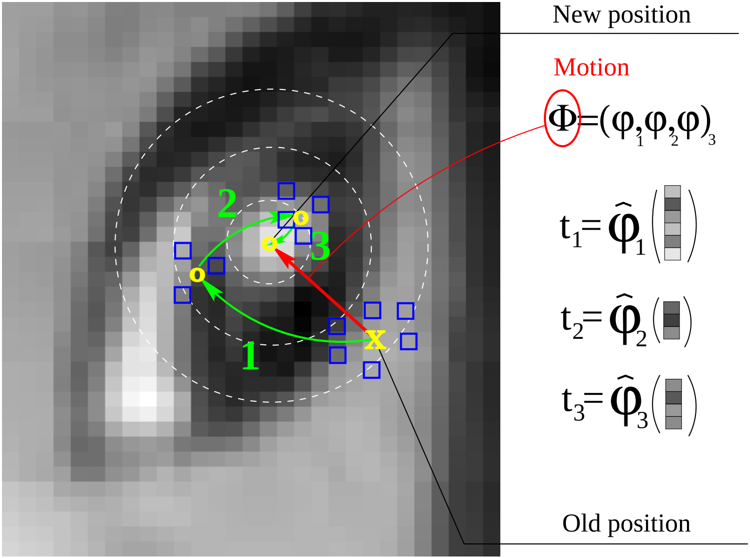

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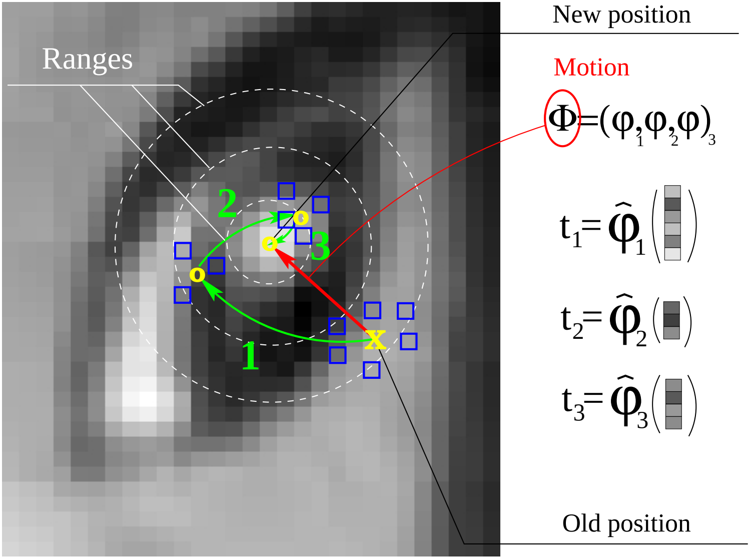

# Learning of sequential predictor

- **Learning** - searching for the sequence with predefined *range*, *accuracy* and minimal *computational cost*.
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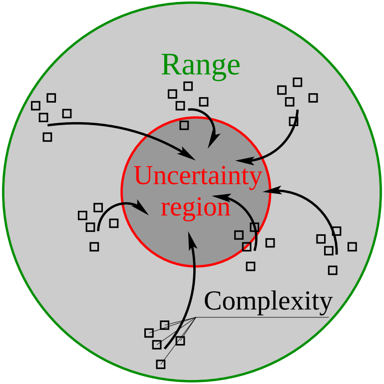


- **Range**: the set of admissible motions, $r$.
- **Complexity**: cardinality of support set, $c$.
- **Uncertainty region**: the region within which all predictions lie, $\lambda$. Small red circles show acceptable uncertainty.

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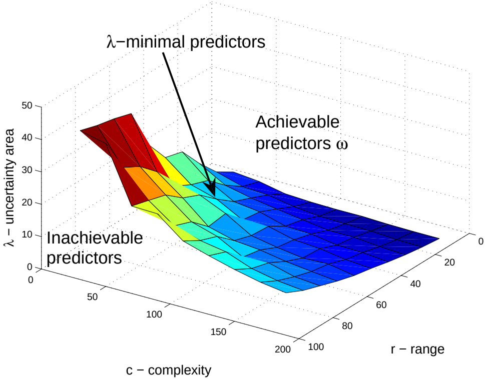


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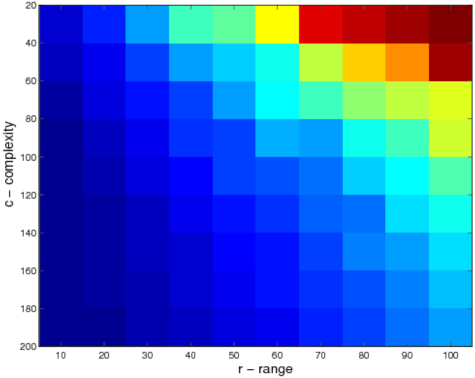


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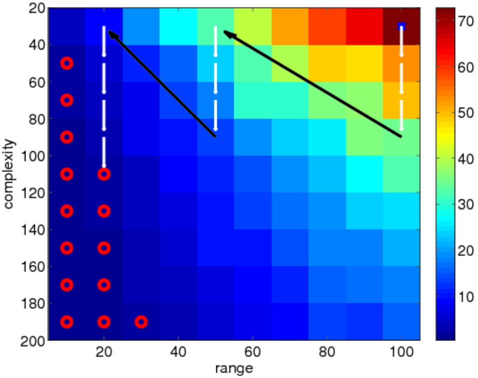


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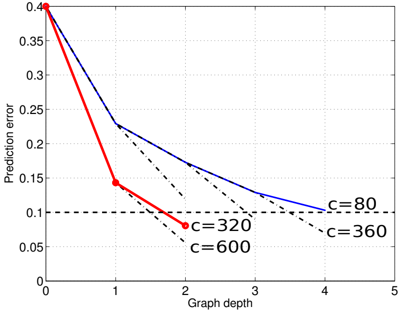


Don't forget to show the live demo!

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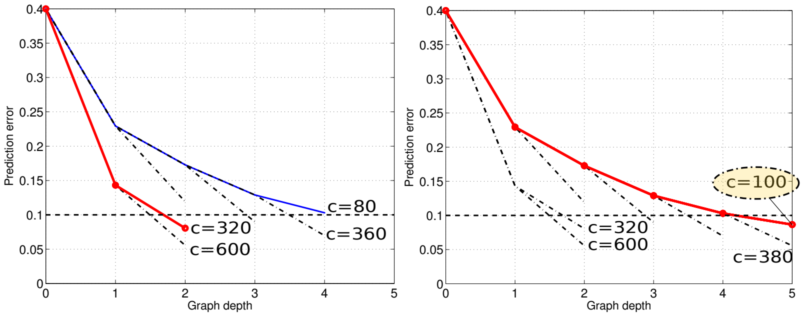


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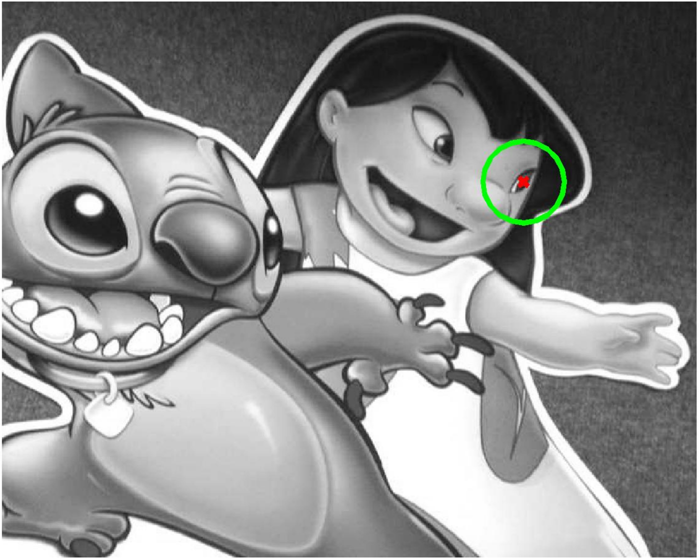
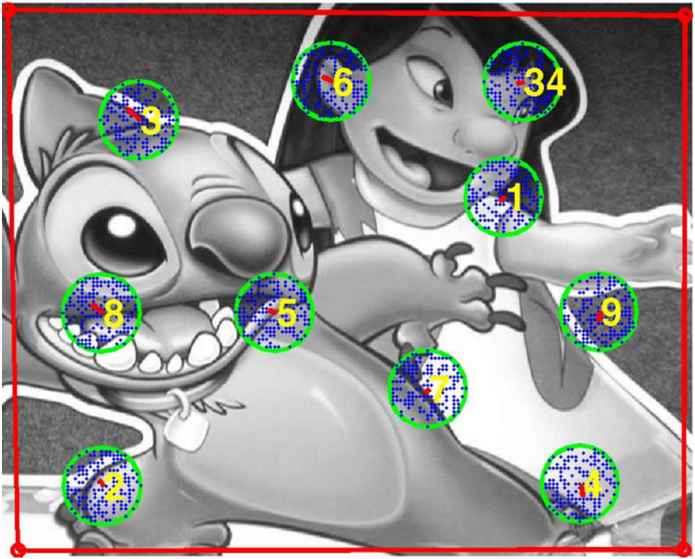


# Modeling motion by number of linear predictors.



# Motion blur, fast motion, views from acute angles and other image distortions.