

Non-Bayesian Methods

lecturer: Jiří Matas, matas@cmp.felk.cvut.cz

authors: Václav Hlaváč, Jiří Matas, Boris Flach, Ondřej Drbohlav

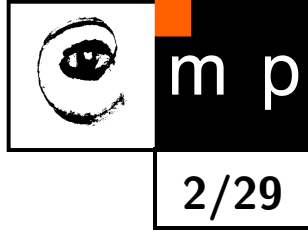
Czech Technical University, Faculty of Electrical Engineering
Department of Cybernetics, Center for Machine Perception
121 35 Praha 2, Karlovo nám. 13, Czech Republic

<http://cmp.felk.cvut.cz>

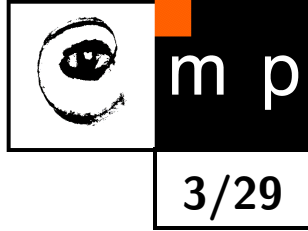
10/Oct/2016

Lecture Outline

1. Limitations of Bayesian Decision Theory
2. Neyman Pearson Task
3. Minimax Task
4. Wald Task



Bayesian Decision Theory



Recall:

X set of observations

K set of hidden states

D set of decisions

p_{XK} : $X \times K \rightarrow \mathbb{R}$: joint probability

W : $K \times D \rightarrow \mathbb{R}$: *loss function*,

q : $X \rightarrow D$: strategy

$R(q)$: risk:

$$R(q) = \sum_{x \in X} \sum_{k \in K} p_{XK}(x, k) W(k, q(x)) \quad (1)$$

Bayesian strategy q^* :

$$q^* = \operatorname{argmin}_{q \in X \rightarrow D} R(q) \quad (2)$$

Limitations of the Bayesian Decision Theory

The limitations follow from the very ingredients of the Bayesian Decision Theory — the necessity to know all the probabilities and the loss function.

- ◆ The loss function W must make sense, but in many tasks it wouldn't
 - medical diagnosis task (W : price of medicines, staff labor, etc. but what penalty in case of patient's death?) Uncomparable penalties on different axes of X .
 - nuclear plant
 - judicial error
- ◆ The prior probabilities $p_K(k)$: must exist and be known. But in some cases it does not make sense to talk about probabilities because the events are not random.
 - $K = \{1, 2\} \equiv \{\text{own army plane, enemy plane}\}$;
 $p(x|1)$, $p(x|2)$ do exist and can be estimated, but $p(1)$ and $p(2)$ don't.
- ◆ The conditionals may be subject to non-random intervention; $p(x | k, z)$ where $z \in Z = \{1, 2, 3\}$ are different interventions.
 - a system for handwriting recognition: The training set has been prepared by 3 different persons. But the test set has been constructed by one of the 3 persons only. This **cannot** be done:

$$(!) \quad p(x | k) = \sum_z p(z)p(x | k, z) \quad (3)$$

Neyman Pearson Task

- ◆ $K = \{D, N\}$ (dangerous state, normal state)
- ◆ X set of observations
- ◆ Conditionals $p(x | D)$, $p(x | N)$ are given
- ◆ The priors $p(D)$ and $p(N)$ are unknown or do not exist
- ◆ $q: X \rightarrow K$ strategy

The Neyman Person Task looks for the optimal strategy q^* for which

- i) the error of classification of the dangerous state is lower than a predefined threshold $\bar{\epsilon}_D$ ($0 < \bar{\epsilon}_D < 1$), while
- ii) the classification error for the normal state is as low as possible.

This is formulated as an optimization task with an inequality constraint:

$$q^* = \operatorname{argmin}_{q: X \rightarrow K} \sum_{x: q(x) \neq N} p(x | N) \quad (4)$$

$$\text{subject to: } \sum_{x: q(x) \neq D} p(x | D) \leq \bar{\epsilon}_D. \quad (5)$$

Neyman Pearson Task

(copied from the previous slide:)

$$q^* = \operatorname{argmin}_{q: X \rightarrow K} \sum_{x: q(x) \neq N} p(x | N) \quad (4)$$

$$\text{subject to: } \sum_{x: q(x) \neq D} p(x | D) \leq \bar{\epsilon}_D. \quad (5)$$

A strategy is characterized by the classification error values ϵ_N and ϵ_D :

$$\epsilon_N = \sum_{x: q(x) \neq N} p(x | N) \quad (\text{false alarm}) \quad (6)$$

$$\epsilon_D = \sum_{x: q(x) \neq D} p(x | D) \quad (\text{overlooked danger}) \quad (7)$$

Example: Male/Female Recognition (Neyman Pearson) (1)

An aging student at CTU wants to marry. He can't afford to miss recognizing a girl when he meets her, therefore he sets the threshold on female classification error to $\bar{\epsilon}_D = 0.2$. At the same time, he wants to minimize mis-classifying boys for girls.

- ◆ $K = \{D, N\} \equiv \{F, M\}$ (female, male)
- ◆ measurements $X = \{\text{short, normal, tall}\} \times \{\text{ultralight, light, avg, heavy}\}$
- ◆ Prior probabilities do not exist.
- ◆ Conditionals are given as follows:

$p(x|F)$

short	.197	.145	.094	.017
normal	.077	.299	.145	.017
tall	.001	.008	.000	.000
	u-light	light	avg	heavy

$p(x|M)$

short	.011	.005	.011	.011
normal	.005	.071	.408	.038
tall	.002	.014	.255	.169
	u-light	light	avg	heavy

(8)

Neyman Pearson : Solution

The optimal strategy q^* for a given $x \in X$ is constructed using the likelihood ratio $\frac{p(x | \text{N})}{p(x | \text{D})}$.

Let there be a constant $\mu \geq 0$. Given this μ , a strategy q is constructed as follows:

$$\frac{p(x | \text{N})}{p(x | \text{D})} > \mu \quad \Rightarrow \quad q(x) = \text{N}, \quad (9)$$

$$\frac{p(x | \text{N})}{p(x | \text{D})} \leq \mu \quad \Rightarrow \quad q(x) = \text{D}. \quad (10)$$

The optimal strategy q^* is obtained by selecting the minimal μ for which there still holds that $\epsilon_{\text{D}} \leq \bar{\epsilon}_{\text{D}}$.

Let us show this on an example.

Example: Male/Female Recognition (Neyman Pearson) (2)

 $p(x|F)$

short	.197	.145	.094	.017
normal	.077	.299	.145	.017
tall	.001	.008	.000	.000
	u-light	light	avg	heavy

 $p(x|M)$

short	.011	.005	.011	.011
normal	.005	.071	.408	.038
tall	.002	.014	.255	.169
	u-light	light	avg	heavy

 $r(x) = p(x|M)/p(x|F)$

short	0.056	0.034	0.117	0.647
normal	0.065	0.237	2.814	2.235
tall	2.000	1.750	∞	∞
	u-light	light	avg	heavy

 rank order of $p(x|M)/p(x|F)$

short	2	1	4	6
normal	3	5	10	9
tall	8	7	11	12
	u-light	light	avg	heavy

Here, different μ 's can produce 11 different strategies.

First, let us take $2.814 < \mu < \infty$, e.g. $\mu = 3$. This produces a strategy $q^*(x) = F$ everywhere except where $p(x|F) = 0$. Obviously, classification error ϵ_F for F is $\epsilon_F = 0$, and $\epsilon_M = 1 - .255 - .169 = .576$.

Example: Male/Female Recognition (Neyman Pearson) (3)

 $p(x|F)$

short	.197	.145	.094	.017
normal	.077	.299	.145	.017
tall	.001	.008	.000	.000
	u-light	light	avg	heavy

 $p(x|M)$

short	.011	.005	.011	.011
normal	.005	.071	.408	.038
tall	.002	.014	.255	.169
	u-light	light	avg	heavy

 $r(x) = p(x|M)/p(x|F)$

short	0.056	0.034	0.117	0.647
normal	0.065	0.237	2.814	2.235
tall	2.000	1.750	∞	∞
	u-light	light	avg	heavy

 rank, and $q^*(x) = \{F, M\}$ for $\mu = 2.5$

short	2	1	4	6
normal	3	5	10	9
tall	8	7	11	12
	u-light	light	avg	heavy

Next, take μ which satisfies

$$r_9 < \mu < r_{10} \quad (\text{e.g. } \mu = 2.5) \tag{11}$$

(where r_i is the likelihood ratios indexed by its rank.)

Here, $\epsilon_F = .145$, and $\epsilon_M = 1 - .255 - .169 - .408 = .168$.

Example: Male/Female Recognition (Neyman Pearson) (4)

 $p(x|F)$

short	.197	.145	.094	.017
normal	.077	.299	.145	.017
tall	.001	.008	.000	.000
	u-light	light	avg	heavy

 $p(x|M)$

short	.011	.005	.011	.011
normal	.005	.071	.408	.038
tall	.002	.014	.255	.169
	u-light	light	avg	heavy

 $r(x) = p(x|M)/p(x|F)$

short	0.056	0.034	0.117	0.647
normal	0.065	0.237	2.814	2.235
tall	2.000	1.750	∞	∞
	u-light	light	avg	heavy

 rank, and $q^*(x) = \{F, M\}$ for $\mu = 2.1$

short	2	1	4	6
normal	3	5	10	9
tall	8	7	11	12
	u-light	light	avg	heavy

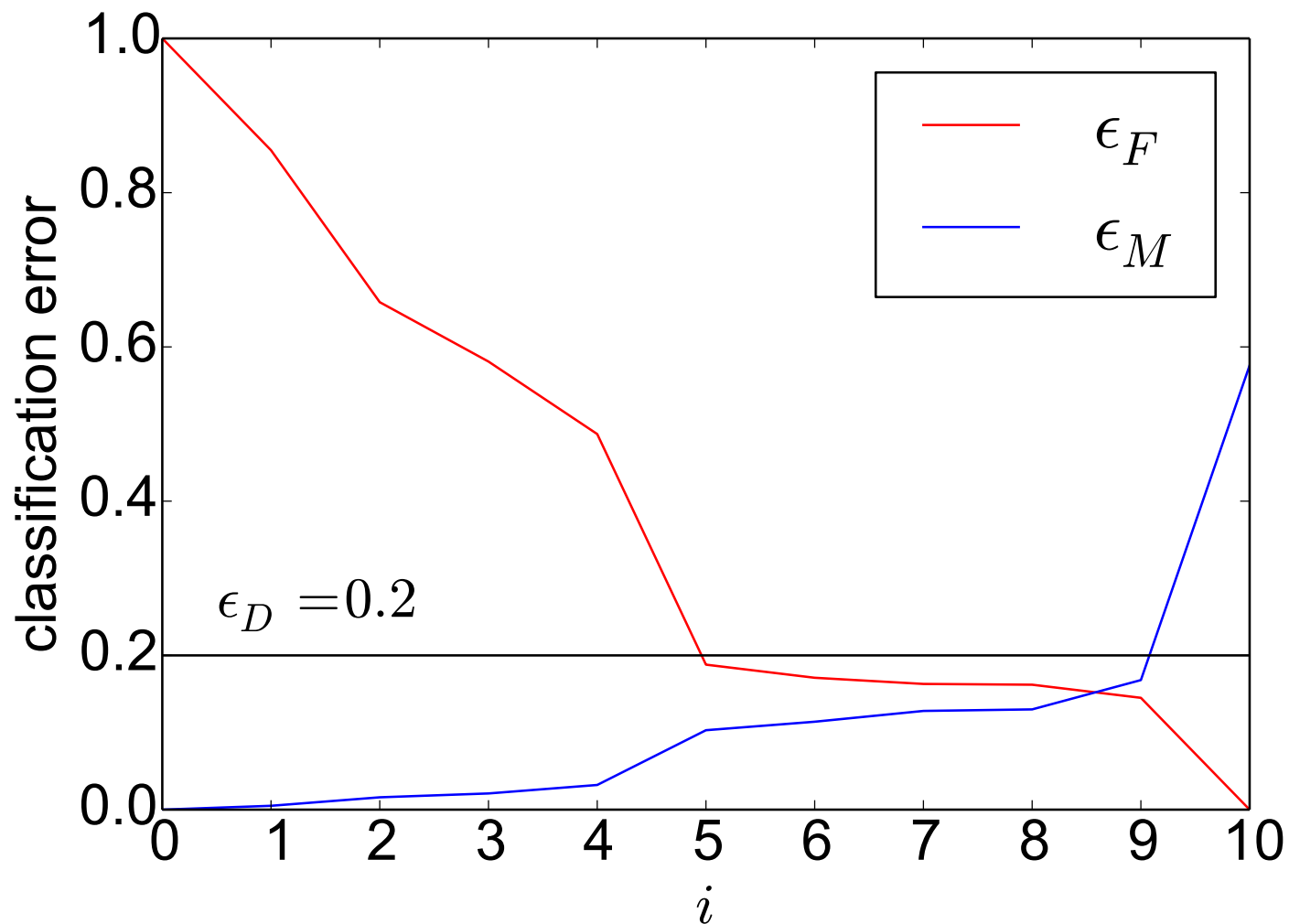
Do the same for μ satisfying

$$r_8 < \mu < r_9 \quad (\text{e.g. } \mu = 2.1) \quad (12)$$

$\Rightarrow \epsilon_F = .162$, and $\epsilon_M = 0.13$.

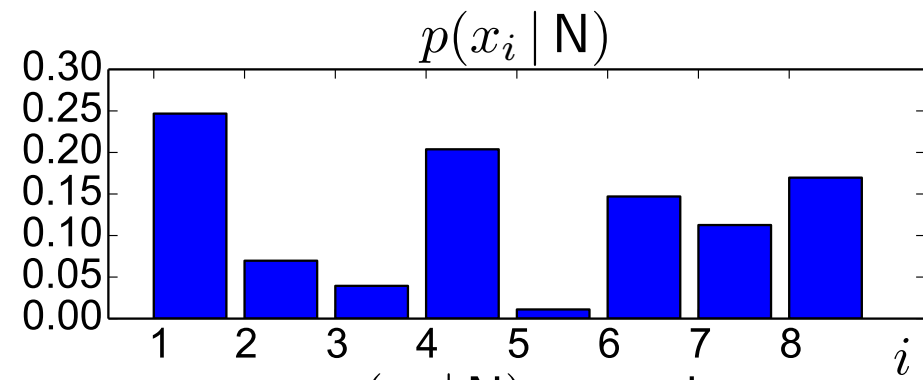
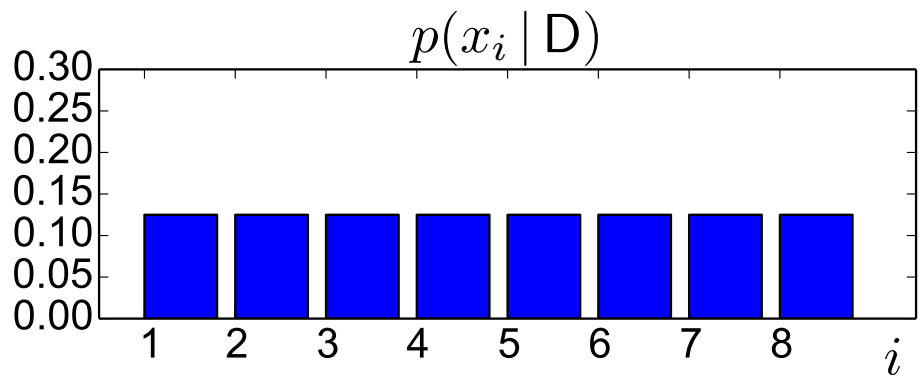
Example: Male/Female Recognition (Neyman Pearson) (5)

Classification errors for F and M, for $\mu_i = \frac{r_i+r_{i+1}}{2}$ and $\mu_0 = 0$.

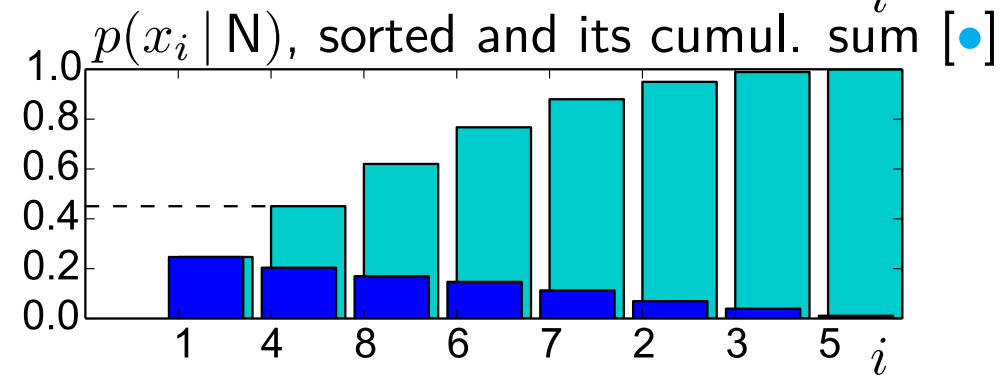
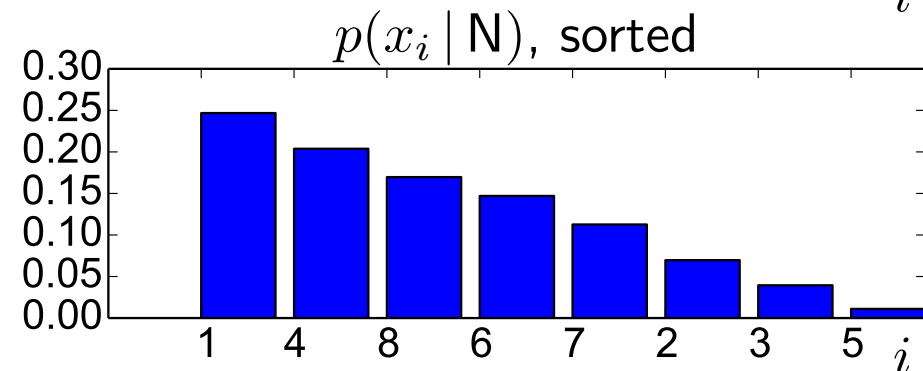


The optimum is reached for $r_5 < \mu < r_6$; $\epsilon_F = .188$, $\epsilon_M = .103$

Neyman Pearson : Solution (1, special case)



Consider first a special case when $p(x_i | D) = \text{const} = \frac{1}{8}$. Possible values for ϵ_D are $0, \frac{1}{8}, \frac{2}{8}, \dots, 1$. If a strategy q classifies P observations as normal then $\epsilon_D = \frac{P}{8}$. Let $P = 1$ and thus $\epsilon_D = \frac{1}{8}$. It is clear that ϵ_N will attain minimum if the (one) observation which is classified as normal is the one with the highest $p(x_i | N)$. Similarly, if $P = 2$ then the two observations to be classified as normal are the one with the first two highest $p(x_i | N)$. Etc.



↑ cumulative sum of sorted $p(x_i | N)$ shows the classification success rate for N, that is, $1 - \epsilon_N$, for $\epsilon_D = \frac{1}{8}, \frac{2}{8}, \dots, 1$. For example, for $\epsilon_D = \frac{2}{8}$ ($P = 2$), $\epsilon_N = 1 - 0.45 = 0.55$ (as shown, dashed.)

Neyman Pearson : Solution (2, general case)

In general, $p(x_i | D) \neq \text{const.}$ Consider the following example:

x_1	x_2	x_3
0.5	0.25	0.25

x_1	x_2	x_3
0.6	0.35	0.05

But this can easily be converted to the previous special case by (only formally) splitting x_1 to two observations x'_1 and x''_1 :

x'_1	x''_1	x_2	x_3
0.25	0.25	0.25	0.25

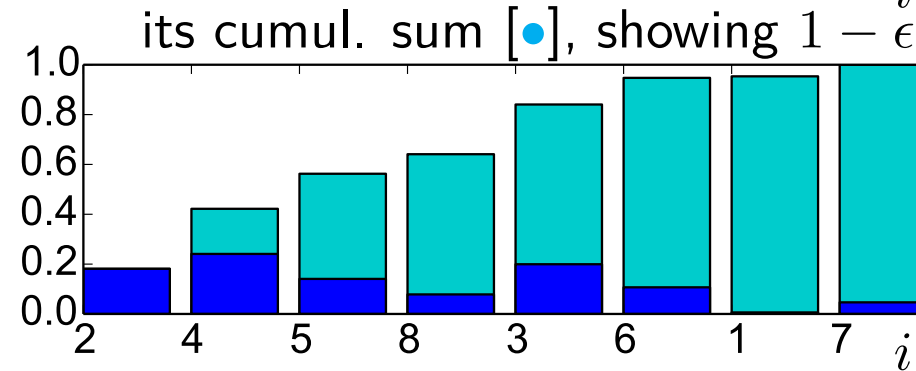
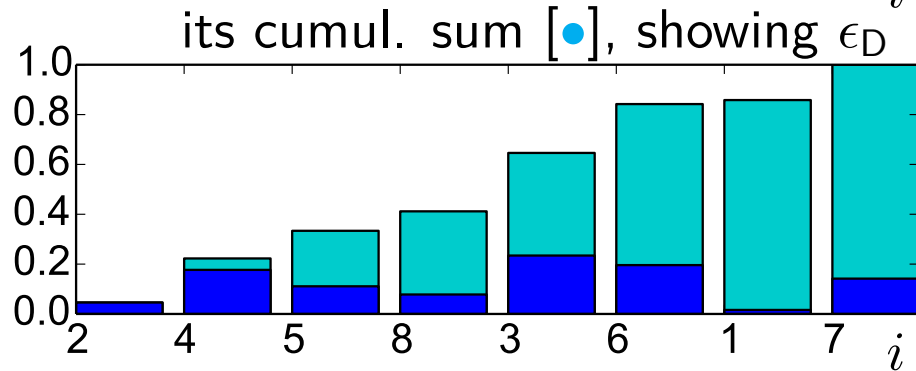
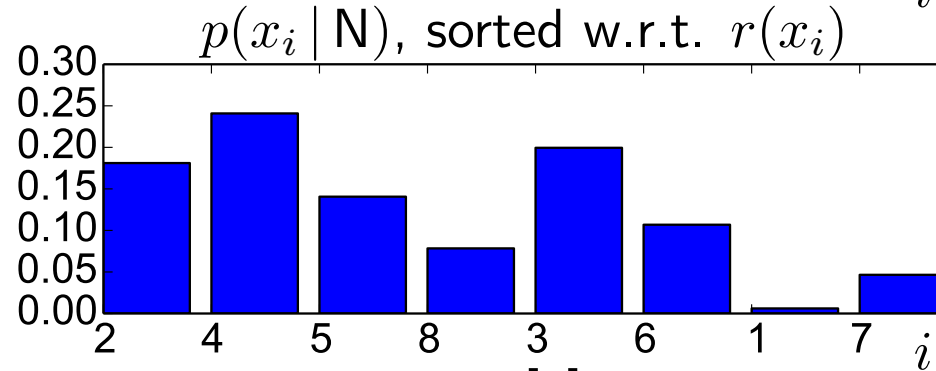
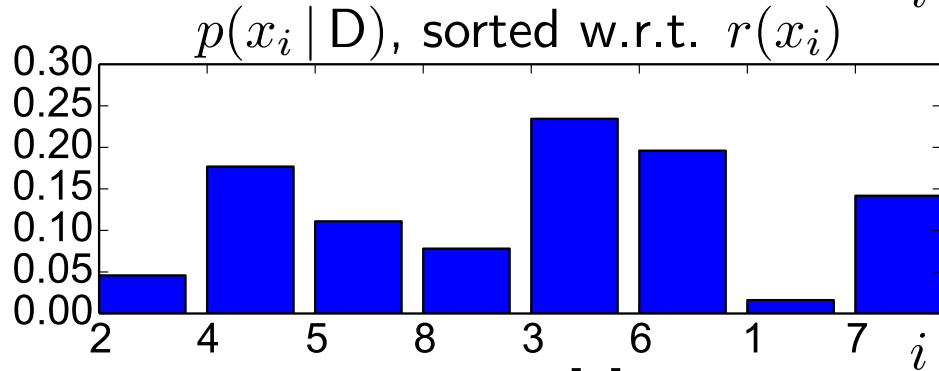
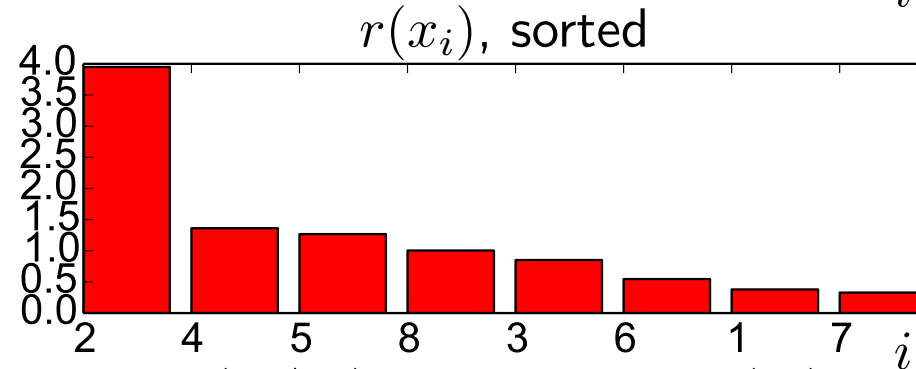
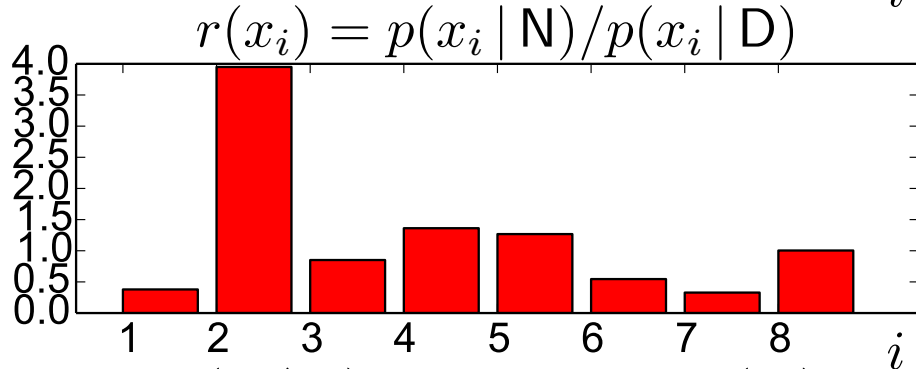
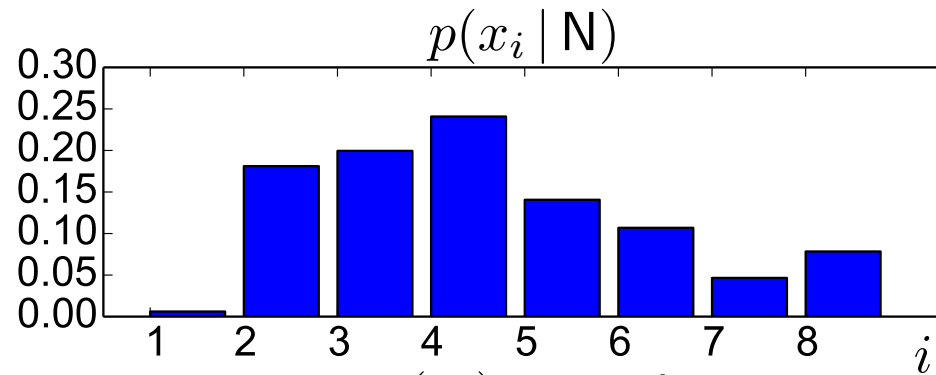
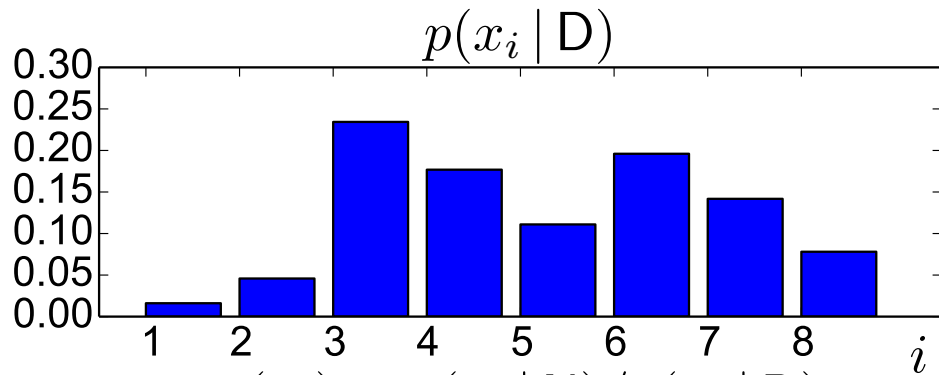
x'_1	x''_1	x_2	x_3
0.3	0.3	0.35	0.05

which would result in ordering the observations by decreasing $p(x_i | N)$ as: x_2, x_1, x_3 .

Obviously, the same effect on ordering is obtained when $p(x_i | N)$ is divided by $p(x_i | D)$, that is, using the likelihood ratio

$$r(x_i) = \frac{p(x_i | N)}{p(x_i | D)}. \tag{13}$$

Neyman Pearson : Solution (3, general case, example)



Neyman Pearson Solution : Illustration of Principle

Lagrangian of the Neyman Pearson Task is

$$L(q) = \underbrace{\sum_{x: q(x)=D} p(x | N)}_{=} + \mu \left(\sum_{x: q(x)=N} p(x | D) - \bar{\epsilon}_D \right) \quad (14)$$

$$= 1 - \sum_{x: q(x)=N} p(x | N) + \mu \left(\sum_{x: q(x)=N} p(x | D) \right) - \mu \bar{\epsilon}_D \quad (15)$$

$$= 1 - \mu \bar{\epsilon}_D + \sum_{x: q(x)=N} \underbrace{\{\mu p(x | D) - p(x | N)\}}_{T(x)} \quad (16)$$

If $T(x)$ is negative for an x then it will decrease the objective function and the optimal strategy q^* will decide $q^*(x) = N$. This illustrates why the solution to the Neyman Pearson Task has the form

$$\frac{p(x | N)}{p(x | D)} > \mu \quad \Rightarrow \quad q(x) = N, \quad (9)$$

$$\frac{p(x | N)}{p(x | D)} \leq \mu \quad \Rightarrow \quad q(x) = D. \quad (10)$$

Neyman Pearson : Derivation (1)

$$q^* = \min_{q: X \rightarrow K} \sum_{x: q(x) \neq N} p(x | N) \quad \text{subject to:} \quad \sum_{x: q(x) \neq D} p(x | D) \leq \bar{\epsilon}_D. \quad (17)$$

Let us rewrite this as

$$q^* = \min_{q: X \rightarrow K} \sum_{x \in X} \alpha(x) p(x | N) \quad \text{subject to:} \quad \sum_{x \in X} [1 - \alpha(x)] p(x | D) \leq \bar{\epsilon}_D. \quad (18)$$

$$\text{and:} \quad \alpha(x) \in \{0, 1\} \quad \forall x \in X \quad (19)$$

This is a combinatorial optimization problem. If the relaxation is done from $\alpha(x) \in \{0, 1\}$ to $0 \leq \alpha(x) \leq 1$, this can be solved by **linear programming** (LP). The Lagrangian of this problem with inequality constraints is:

$$L(\alpha(x_1), \alpha(x_2), \dots, \alpha(x_N)) = \sum_{x \in X} \alpha(x) p(x | N) + \mu \left(\sum_{x \in X} [1 - \alpha(x)] p(x | D) - \bar{\epsilon}_D \right) \quad (20)$$

$$- \sum_{x \in X} \mu_0(x) \alpha(x) + \sum_{x \in X} \mu_1(x) (\alpha(x) - 1) \quad (21)$$

Neyman Pearson : Derivation (2)

$$L(\alpha(x_1), \alpha(x_2), \dots, \alpha(x_N)) = \sum_{x \in X} \alpha(x)p(x | \mathbf{N}) + \mu \left(\sum_{x \in X} [1 - \alpha(x)]p(x | \mathbf{D}) - \bar{\epsilon}_D \right) \quad (20)$$

$$- \sum_{x \in X} \mu_0(x)\alpha(x) + \sum_{x \in X} \mu_1(x)(\alpha(x) - 1) \quad (21)$$

The conditions for optimality are ($\forall x \in X$):

$$\frac{\partial L}{\partial \alpha(x)} = p(x | \mathbf{N}) - \mu p(x | \mathbf{D}) - \mu_0(x) + \mu_1(x) = 0, \quad (22)$$

$$\mu \geq 0, \mu_0(x) \geq 0, \mu_1(x) \geq 0, \quad 0 \leq \alpha(x) \leq 1, \quad (23)$$

$$\mu_0(x)\alpha(x) = 0, \mu_1(x)(\alpha(x) - 1) = 0, \mu \left(\sum_{x \in X} [1 - \alpha(x)]p(x | \mathbf{D}) - \bar{\epsilon}_D \right) = 0. \quad (24)$$

Case-by-case analysis:

case	implications
$\mu = 0$	L minimized by $\alpha(x) = 0 \quad \forall x$
$\mu \neq 0, \alpha(x) = 0$	$\mu_1(x) = 0 \Rightarrow \mu_0(x) = p(x \mathbf{N}) - \mu p(x \mathbf{D}) \Rightarrow p(x \mathbf{N})/p(x \mathbf{D}) \leq \mu$
$\mu \neq 0, \alpha(x) = 1$	$\mu_0(x) = 0 \Rightarrow \mu_1(x) = -[p(x \mathbf{N}) - \mu p(x \mathbf{D})] \Rightarrow p(x \mathbf{N})/p(x \mathbf{D}) \geq \mu$
$\mu \neq 0,$ $0 < \alpha(x) < 1$	$\mu_0(x) = \mu_1(x) = 0 \Rightarrow p(x \mathbf{N})/p(x \mathbf{D}) = \mu$

Neyman Pearson : Derivation (3)

Case-by-case analysis:

case	implications
$\mu = 0$	L minimized by $\alpha(x) = 0 \quad \forall x$
$\mu \neq 0, \alpha(x) = 0$	$\mu_1(x) = 0 \Rightarrow \mu_0(x) = p(x \mathbf{N}) - \mu p(x \mathbf{D}) \Rightarrow p(x \mathbf{N})/p(x \mathbf{D}) \leq \mu$
$\mu \neq 0, \alpha(x) = 1$	$\mu_0(x) = 0 \Rightarrow \mu_1(x) = -[p(x \mathbf{N}) - \mu p(x \mathbf{D})] \Rightarrow p(x \mathbf{N})/p(x \mathbf{D}) \geq \mu$
$\mu \neq 0,$ $0 < \alpha(x) < 1$	$\mu_0(x) = \mu_1(x) = 0 \Rightarrow p(x \mathbf{N})/p(x \mathbf{D}) = \mu$

Optimal Strategy for a given $\mu \geq 0$ and particular $x \in X$:

$$\frac{p(x | \mathbf{N})}{p(x | \mathbf{D})} \begin{cases} < \mu & \Rightarrow q(x) = \mathbf{D} \text{ (as } \alpha(x) = 0) \\ > \mu & \Rightarrow q(x) = \mathbf{N} \text{ (as } \alpha(x) = 1) \\ = \mu & \Rightarrow \text{LP relaxation does not give the desired solution, as } \alpha \notin \{0, 1\} \end{cases} \quad (25)$$

Neyman Pearson : Note on Randomized Strategies (1)

Consider:

$p(x D)$		
x_1	x_2	x_3
0.9	0.09	0.01

$p(x N)$		
x_1	x_2	x_3
0.09	0.9	0.01

$r(x) = p(x N)/p(x D)$		
x_1	x_2	x_3
0.1	10	1

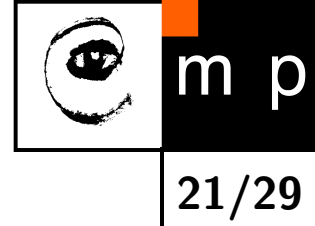
and $\bar{\epsilon}_D = 0.03$.

- ◆ $q_1 : (x_1, x_2, x_3) \rightarrow (D, D, D) \Rightarrow \epsilon_D = 0.00, \epsilon_N = 1.00$
- ◆ $q_2 : (x_1, x_2, x_3) \rightarrow (D, D, N) \Rightarrow \epsilon_D = 0.01, \epsilon_N = 0.99$
- ◆ no other deterministic strategy q is feasible, that is all other ones have $\epsilon_D > \bar{\epsilon}_D$
- ◆ q_2 is the best deterministic strategy but it does not comply with the previous basic result of constructing the optimal strategy because it decides for N for likelihood ratio 1 but decides for D for likelihood ratios 0.01 and 10. Why is that?
- ◆ we can construct a randomized strategy which attains $\bar{\epsilon}_D$ and reaches lower ϵ_N :

$$q(x_1) = q(x_3) = D, \quad q(x_2) = \begin{cases} N & 1/3 \text{ of the time} \\ D & 2/3 \text{ of the time} \end{cases} \quad (26)$$

For such strategy, $\epsilon_D = 0.03, \epsilon_N = 0.7$.

Neyman Pearson : Note on Randomized Strategies (2)



- ◆ This is not a problem but a feature which is caused by discrete nature of X (does not happen when X is continuous).
- ◆ This is exactly what the case of $\mu = p(x | N)/p(x | D)$ is on slide 18.

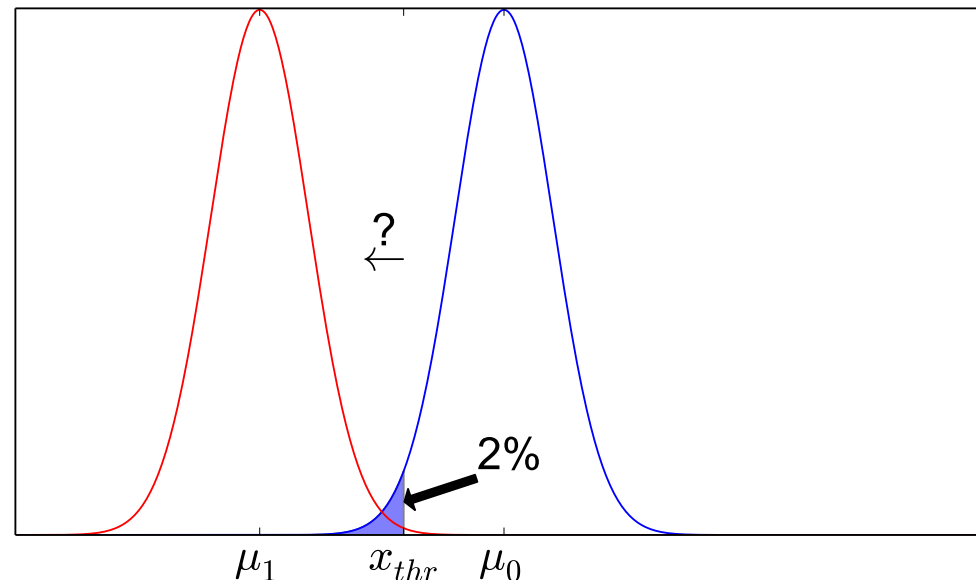
Neyman Pearson : Notes (1)

- ◆ The task can be generalized to 3 hidden states, of which 2 are dangerous, $K = \{N, D_1, D_2\}$. It is formulated as an analogous problem with two inequality constraints and minimization of classification error for N.
- ◆ Neyman's and Pearson's work dates to 1928 and 1933.
- ◆ A particular strength of the approach lies in that the likelihood ratio $r(x)$ or even $p(x | N)$ need not be known. For the task to be solved, it is enough to know the $p(x | D)$ and the **rank order** of the likelihood ratio (to be demonstrated on the next page)

Neyman Pearson : Notes (2)

- ◆ Consider a medicine for reducing weight. The normal population has a distribution of weight $p(x | D)$ as shown in blue. Let it be normal, $p(x | D) = \mathcal{N}(x | \mu_0, \sigma)$. The distribution of weights after 1 month of taking the medicine is assumed to be normal as well, with the same variance but unknown shift of mean to the left, $p(x | N) = \mathcal{N}(x | \mu_1, \sigma)$, with $\mu_1 < \mu_0$ but otherwise unknown (shown in red). The likelihood ratio is

$$r(x) = \exp \frac{1}{2\sigma^2} (-(x - \mu_1)^2 + (x - \mu_0)^2) = \exp \left(\frac{1}{\sigma^2} (\mu_1 - \mu_0)x + \text{const} \right)$$
.
 It is thus decreasing (monotone) with x (irrespective of μ_1 , $\mu_1 < \mu_0$).
- ◆ Setting $\bar{\epsilon}_D = 0.02$, we go along the decreasing $r(x)$ and find the point x_{thr} for which $\int_{-\infty}^{x_{thr}} p(x | D) = \bar{\epsilon}_D = 0.02$ (0.02-quantile). Note that the threshold μ on $r(x)$ is still unknown as $p(x | N)$ is unknown.



Minimax Task

- ◆ $K = \{1, 2, \dots, N\}$
- ◆ X set of observations
- ◆ Conditionals $p(x | k)$ are known $\forall k \in K$
- ◆ The priors $p(k)$ are unknown or do not exist
- ◆ $q: X \rightarrow K$ strategy

The Minimax Task looks for the optimum strategy q^* which minimizes the classification error of the worst classified class:

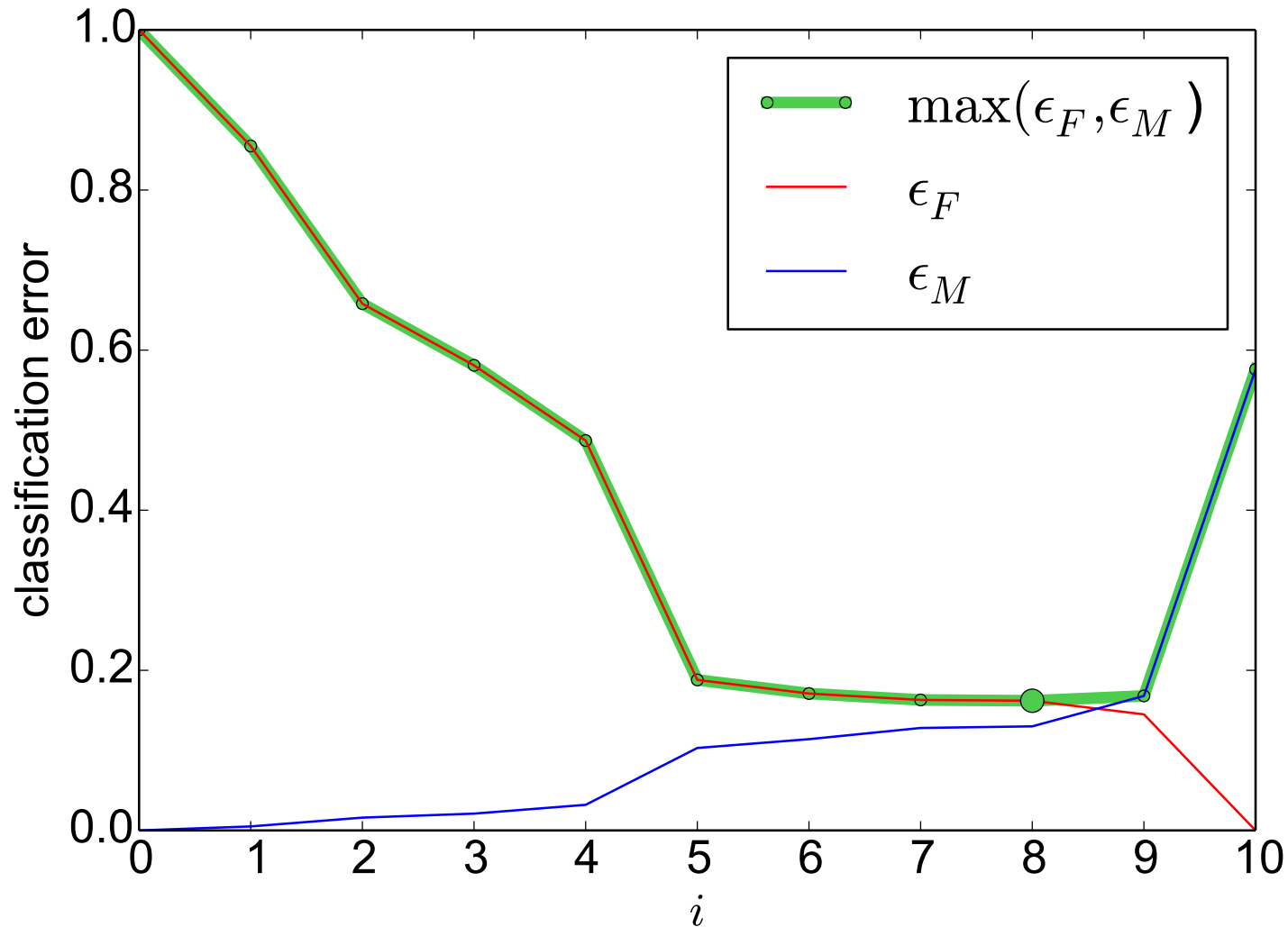
$$q^* = \operatorname{argmin}_{q: X \rightarrow K} \max_{k \in K} \epsilon(k), \quad \text{where} \quad (27)$$

$$\epsilon(k) = \sum_{x: q(x) \neq k} p(x | k) \quad (28)$$

- ◆ Example: A recognition algorithm qualifies for a competition using preliminary tests. During the final competition, only objects from the hardest-to-classify class are used.
- ◆ For a 2-class problem, the strategy is again constructed using the likelihood ratio.
- ◆ In the case of continuous observations space X , equality of classification errors is attained: $\epsilon_1 = \epsilon_2$
- ◆ The derivation can again be done using Linear Programming.

Example: Male/Female Recognition (Minimax)

Classification errors for F and M, for $\mu_i = \frac{r_i+r_{i+1}}{2}$ and $\mu_0 = 0$.



The optimum is attained for $i = 8$, $\epsilon_F = .162$, $\epsilon_M = .13$. The corresponding strategy is as shown on slide [11](#).

Minimax: Comparison with Bayesian Decision with Unknown Priors

- ◆ Consider the same setting as in the Minimax task, but let the priors $p(k)$ exist but be unknown.
- ◆ The Bayesian error ϵ for strategy q is

$$\epsilon = \sum_k \sum_{x: q(x) \neq k} p(x, k) = \sum_k p(k) \underbrace{\sum_{x: q(x) \neq k} p(x | k)}_{\epsilon(k)} \quad (29)$$

- ◆ We want to minimize ϵ but we do not know $p(k)$'s. What is the maximum it can attain? Obviously, the $p(k)$'s do the convex combination of the class errors $\epsilon(k)$; the maximum Bayesian error will be attained when $p(k) = 1$ for the class k with the highest class error $\epsilon(k)$.
- ◆ Thus, to minimize the Bayesian error ϵ under this setting, the solution is to minimize the error of the hardest-to-classify class.
- ◆ Therefore, Minimax formulation and the Bayesian formulation with Unknown Priors lead to the same solution.

Wald Task (1)

- ◆ Let us consider classification with two states, $K = \{1, 2\}$.
- ◆ We want to set a threshold ϵ on the classification error of both of the classes: $\epsilon_1 \leq \epsilon$, $\epsilon_2 \leq \epsilon$.
- ◆ As the previous analysis shows (Neyman Pearson, Minimax), there may be **no** feasible solution if ϵ is set too low.
- ◆ That is why the possibility of decision “do not know” is introduced. Thus $D = K \cup \{?\}$
- ◆ A strategy $q : X \rightarrow D$ is characterized by:

$$\epsilon_1 = \sum_{x: q(x)=2} p(x | 1) \quad (\text{classification error for 1}) \quad (30)$$

$$\epsilon_2 = \sum_{x: q(x)=1} p(x | 2) \quad (\text{classification error for 2}) \quad (31)$$

$$\kappa_1 = \sum_{x: q(x)=?} p(x | 1) \quad (\text{undecided rate for 1}) \quad (32)$$

$$\kappa_2 = \sum_{x: q(x)=?} p(x | 2) \quad (\text{undecided rate for 2}) \quad (33)$$

Wald Task (2)

- ◆ The optimal strategy q^* :

$$q^* = \operatorname{argmin}_{q: X \rightarrow D} \max_{i=\{1,2\}} \kappa_i \quad (34)$$

$$\text{subject to: } \epsilon_1 \leq \epsilon, \epsilon_2 \leq \epsilon \quad (35)$$

- ◆ The task is again solvable using LP (even for more than 2 classes)
- ◆ The optimal solution is again based on the likelihood ratio

$$r(x) = \frac{p(x | 1)}{p(x | 2)} \quad (36)$$

- ◆ The optimal strategy is constructed using suitably chosen thresholds μ_l and μ_h such that:

$$q(x) = \begin{cases} 2 & \text{for } r(x) < \mu_l \\ 1 & \text{for } r(x) > \mu_h \\ ? & \text{for } \mu_l \leq r(x) \leq \mu_h \end{cases} \quad (37)$$

Example: Male/Female Recognition (Wald)

Solve the Wald task for $\epsilon = 0.05$.

$p(x|F)$

short	.197	.145	.094	.017
normal	.077	.299	.145	.017
tall	.001	.008	.000	.000
	u-light	light	avg	heavy

$p(x|M)$

short	.011	.005	.011	.011
normal	.005	.071	.408	.038
tall	.002	.014	.255	.169
	u-light	light	avg	heavy

$r(x) = p(x|M)/p(x|F)$

short	0.056	0.034	0.117	0.647
normal	0.065	0.237	2.814	2.235
tall	2.000	1.750	∞	∞
	u-light	light	avg	heavy

rank, and $q^*(x) = \{F, M, ?\}$

short	2	1	4	6
normal	3	5	10	9
tall	8	7	11	12
	u-light	light	avg	heavy

Result: $\epsilon_M = 0.032$, $\epsilon_F = 0$, $\kappa_M = 0.544$, $\kappa_F = 0.487$

$(r_4 < \mu_l < r_5, r_{10} < \mu_h < \infty)$