Labs from Optimization
Decomposition by singular values (SVD)

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2015

Excercise 1: Aproximation of points in line

Given an m points in the plane $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \in \mathbb{R}^2$. Find the line in the plane (i.e. affine subspace with dimension 1 of space $\mathbb{R}^2$) such that the sum of squares *orthogonal* distances of points on this line was minimal. Imagine that someone draw the points by mouse in Graphical User Interface (GUI) and your task is to space them with the best line.

Help: Optimum line passes through the center of gravity points.

Tasks which need to be solved:

1. Upload the file

   `http://cmp.felk.cvut.cz/cmp/courses/OPT/cviceni/mocap/data_A.mat`.

   It contains matrix $\mathbf{A} \in \mathbb{R}^{m \times 2}$ with rows $\boldsymbol{a}_1^T, \ldots, \boldsymbol{a}_m^T$.

   Output: Nothing.

2. Display one image with specified points $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ (blue) and their orthogonal projection $\tilde{\boldsymbol{a}}_1, \ldots, \tilde{\boldsymbol{a}}_m$ to found the line (red).

   Output: Graph.

3. What is the sum of the squares of the orthogonal distances of points to the found line?

   Output: Number.

4. Find the desired line in two different representations:

   $$\{\, \boldsymbol{y} \in \mathbb{R}^2 \mid \boldsymbol{y}^T \boldsymbol{x} = \alpha \,\} = \{\, \boldsymbol{y}_0 + t\boldsymbol{s} \mid t \in \mathbb{R} \,\}.$$

   Output of this task will be $\boldsymbol{x} \in \mathbb{R}^2$, $\alpha \in \mathbb{R}$, $\boldsymbol{y}_0 \in \mathbb{R}^2$, $\boldsymbol{s} \in \mathbb{R}^2$. Vectors chosen so that $\|\boldsymbol{x}\| = 1$, $\|\boldsymbol{s}\| = 1$, and norm $\|\boldsymbol{y}_0\|$ was minimal (i.e. point $\boldsymbol{y}_0$ was orthogonal projection in the origin $\mathbf{0}$ on line).

   Output: Vectors $\boldsymbol{x} \in \mathbb{R}^2$, $\boldsymbol{y}_0 \in \mathbb{R}^2$, $\boldsymbol{s} \in \mathbb{R}^2$ and number $\alpha \in \mathbb{R}$.

Excercise 2: Compression sequences of motion capture

For creating computer games or animated films is used technology *motion capture*. On the live actor is fixed targets reflecting infrared light. Targets are attached on the important points on the body, such as joints. The amount of special cameras than scans the position of the disks and calculated position of each target in three dimensions for each image. The position of targets in space are then used for example: character animation synthesized computer graphics. See e.g.: Wikipedia.

To obtain a continuous movement must be scanned at a high frequency. For example, data used in our task was scanned with a sampling rate of 120 Hz. As a result, then you need to work with large volumes of data, which is a disadvantage. Our task is to decrease the amount of data so that we lost as little useful information as possible.

Spatial position of a target in one image is given by a trio of coordinates. In our case, we have $\ell = 41$ disks. Location of all disks in one image is given by the vector $\boldsymbol{a} \in \mathbb{R}^n$ of length $n = 3\ell$. It can be thought of as a point in the n-dimensional space. In total, we have m images, thus vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \in \mathbb{R}^n$.

We are looking for items that the best approximate the original points and also can represent a smaller volume of data. Specifically, we are looking for points of $\tilde{\boldsymbol{a}}_1, \ldots, \tilde{\boldsymbol{a}}_m$, which lies in the affine subspace dimension $r < n$ linear space $\mathbb{R}^n$ while minimizing total squared distances from the original points, ie. number

$$\sum_{i=1}^{m} \|\tilde{\boldsymbol{a}}_i - \boldsymbol{a}_i\|^2. \tag{1}$$

Of the lectures you know the optimal affine subspace passing through the center of gravity of points

$$\boldsymbol{c} = \frac{1}{m} \sum_{i=1}^{m} \boldsymbol{a}_i.$$

So when you move the points so that their focus was in the origin, so instead affine subspace you can search linear subspace.

How works data compression? At first sight it seems that the original and approximated points occupy the same amount of memory ($m \times n$ numbers), so we saved nothing. However, the new points lie in a subspace of lower dimension r which can be represented bases. Lets have this base orthonormal and forms a column of matrix $\tilde{\mathbf{V}} \in \mathbb{R}^{n \times r}$. Then each point in subspace is a linear combination of basis vectors

$$\tilde{\boldsymbol{a}} = \tilde{\mathbf{V}} \boldsymbol{y}, \tag{2}$$

where the components of the vector $\boldsymbol{y} \in \mathbb{R}^r$ coefficients are linear combinations. To save all the points $\tilde{\boldsymbol{a}}_1, \ldots, \tilde{\boldsymbol{a}}_m$ we must save the base vectors ($r \times n$ numbers, which is negligible), and then vectors $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$ ($r \times m$ numbers).

Tasks which need to be solved:

1. Download the files `makarena1.txt` and `walk1.txt` from folder[1]

[1] The sequences were downloaded from the site `http://movlab.ulusofona.pt`.

`http://cmp.felk.cvut.cz/cmp/courses/OPT/cviceni/mocap/`.

These files describe the sequence 'Dancing Macarena' and 'Walking'. Each file contains a matrix of dimension $m \times n$ (mark it as $\mathbf{A}$) with rows $\boldsymbol{a}_1^T, \ldots, \boldsymbol{a}_m^T$. To visualize sequences use a script `show_motion.m` (requires download `connected_points.txt`).

Output: Nothing.

2. Minimize criteria (1) under the condition that points $\tilde{\boldsymbol{a}}_1, \ldots, \tilde{\boldsymbol{a}}_m$ are located in affine subspace dimension r. As a result will be matrix $\tilde{\mathbf{A}}$ with rows $\tilde{\boldsymbol{a}}_1^T, \ldots, \tilde{\boldsymbol{a}}_m^T$. Perform the sequence 'Walking' and for five different ranks $r \in \{1, 2, 5, 10, 15\}$. To check out individual solutions play script `show_motion2.m`, which displays the sequence of matrix $\mathbf{A}$ together with the sequence in the matrix $\tilde{\mathbf{A}}$. If your solution is correct, the two sequences similar.

   Output: Table with optimal values of criteria (1) for a specified rank $r \in \{1, 2, 5, 10, 15\}$.

3. The resulting points express as a linear combination (2) of basis vectors. For $r = 2$ draw a sequence of vetors $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$ as trajectory in the plane (consecutive points to connect segment). Do the same also in the third dimension space, therefore for $r = 3$. Perform the sequence 'Dancing Macarena' and 'Walking'.

   Output: Two graphs showing the 2D trajectory and two graphs showing the 3D trajectory.

4. Consider that the figure makes clear translational motion, ie. configuration of points do not change and their coordinates are moving in a straight line. What is the minimum dimension of the subspace that approximation error was zero?

   Output: Number and a short theoretical justification.

5. Suppose we want to calculate the optimal error approximation boundary (1) for different rank $r \leq n$. The result would therefore be n numbers, the last of which would be zero. How to relate these figures to the singular numbers? As an inspiration to draw a singular, listed in the chart.

   Output: Formula.