CZECH TECHNICAL UNIVERSITY IN PRAGUE

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Linear models for classification.
Perceptron. Logistic regression.

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Linear classification

Binary classification task (dichotomy)

Let's have the training dataset $T = \{(\boldsymbol{x}^{(1)}, y^{(1)}), \ldots, (\boldsymbol{x}^{(|T|)}, y^{(|T|)})\}$:

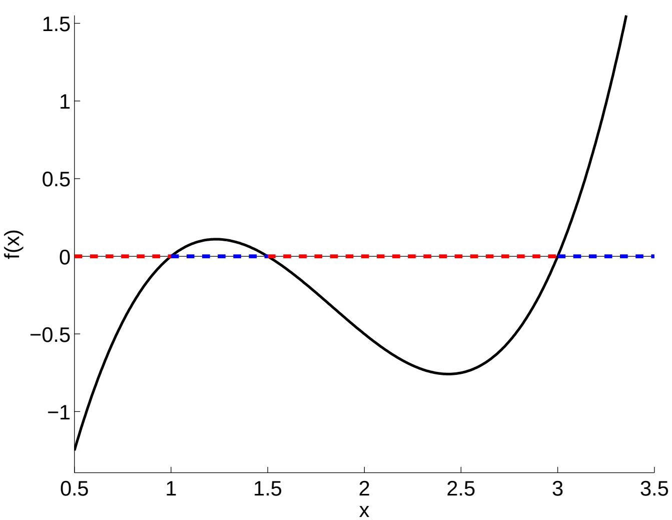
- each example is described by a vector of features $\boldsymbol{x} = (x_1, \ldots, x_D)$,
- each example is labeled with the correct class $y \in \{+1, -1\}$.

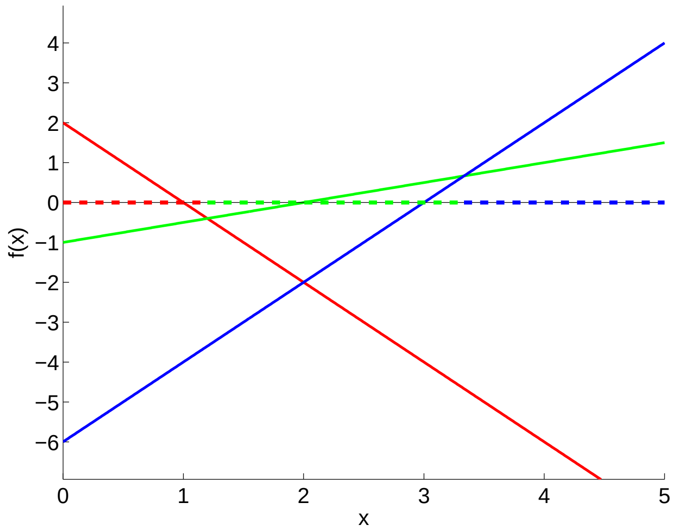
Discrimination function: a function allowing us to *decide* to which class an example $\boldsymbol{x}$ belongs.

- For 2 classes, 1 discrimination function is enough.
- Decision rule:

$$\left.\begin{array}{l} f(\boldsymbol{x}^{(i)}) > 0 \Longleftrightarrow \widehat{y}^{(i)} = +1 \\ f(\boldsymbol{x}^{(i)}) < 0 \Longleftrightarrow \widehat{y}^{(i)} = -1 \end{array}\right\} \qquad \text{i.e.} \qquad \widehat{y}^{(i)} = \operatorname{sign}\left(f(\boldsymbol{x}^{(i)})\right)$$

- *Learning* then amounts to finding (parameters of) function f.





Naive approach

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Naive solution: fit linear regression model to the data!

- Use cost function

$$J_{MSE}(\boldsymbol{w}, T) = \frac{1}{|T|} \sum_{i=1}^{|T|} \left(y^{(i)} - f(\boldsymbol{w}, \boldsymbol{x}^{(i)}) \right)^2,$$

- minimize it with respect to $\boldsymbol{w}$,
- and use $\widehat{y} = \mathrm{sign}(f(\boldsymbol{x}))$.
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Better solution: fit a linear discrimination function which minimizes the number of errors!

- Cost function:

$$J_{01}(\boldsymbol{w}, T) = \frac{1}{|T|}\sum_{i=1}^{|T|}\mathbb{I}(y^{(i)} \neq \widehat{y}^{(i)}),$$

  where $\mathbb{I}$ is the indicator function: $\mathbb{I}(a)$ returns 1 iff a is True, 0 otherwise.
- The cost function is non-smooth, contains plateaus, not easy to optimize, but there are algorithms which attempt to solve it, e.g. perceptron, Kozinec's algorithm, etc.

Perceptron

Perceptron algorithm

Perceptron [Ros62]:

- a simple model of a neuron
- linear classifier (in this case a classifier with linear discrimination function)

Algorithm 1: Perceptron algorith

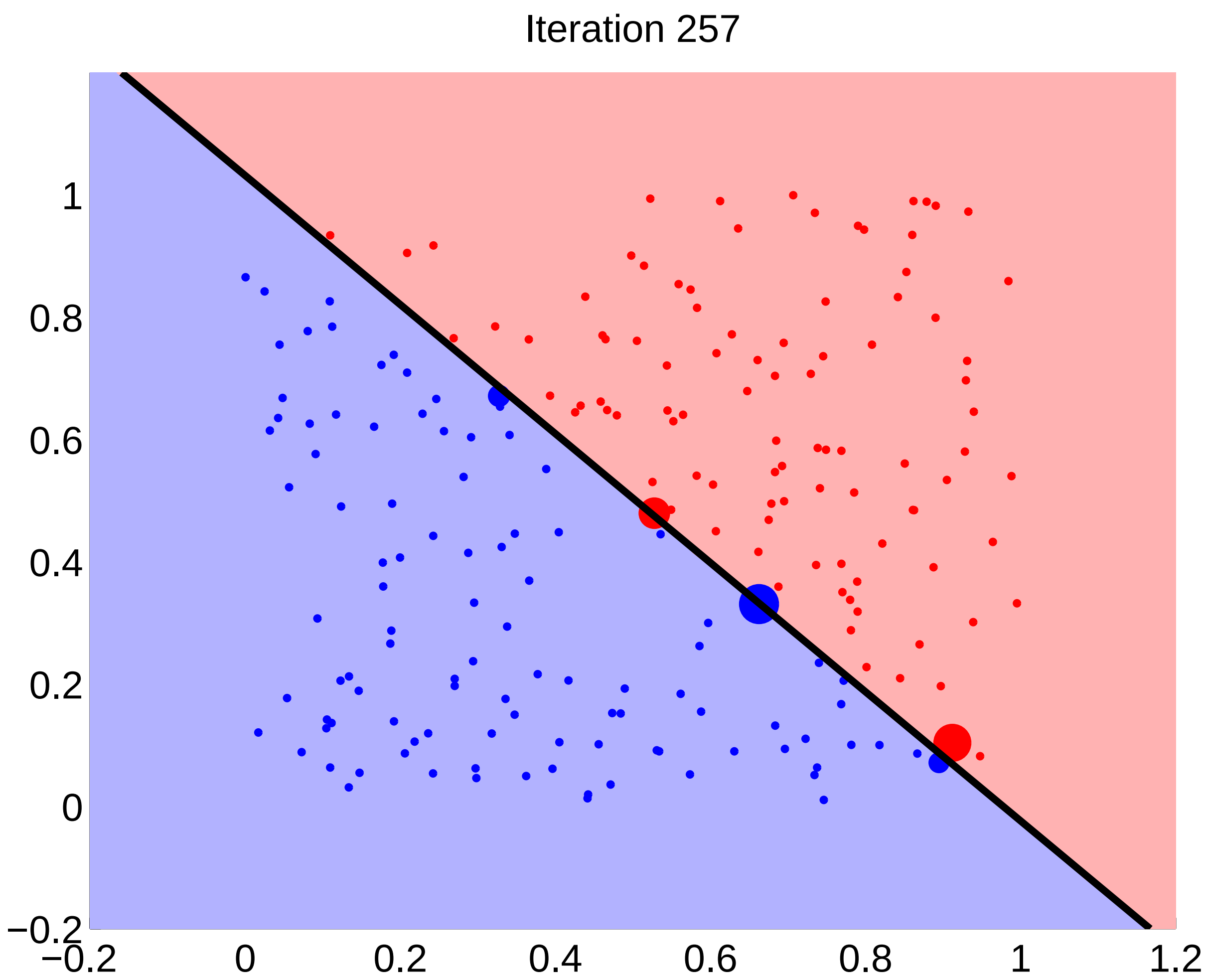
Input: Linearly separable training dataset: $\{\boldsymbol{x}^{(i)}, y^{(i)}\}, \boldsymbol{x}^{(i)} \in \mathcal{R}^{D+1}$ (homogeneous coordinates), $y^{(i)} \in \{+1, -1\}$

Output: Weight vector $\boldsymbol{w}$ such that $\boldsymbol{x}^{(i)}\boldsymbol{w}^T > 0$ iff $y^{(i)} = +1$ and $\boldsymbol{x}^{(i)}\boldsymbol{w}^T < 0$ iff $y^{(i)} = -1$

1 **begin**
2   Initialize the weight vector, e.g. $\boldsymbol{w} = \boldsymbol{0}$.
3   Invert all examples $\boldsymbol{x}$ belonging to class -1: $\boldsymbol{x}^{(i)} = -\boldsymbol{x}^{(i)}$ for all i, where $y^{(i)} = -1$.
4   Find an incorrectly classified training vector, i.e. find j such that $\boldsymbol{x}^{(i)}\boldsymbol{w}^T \leq 0$, e.g. the worst classified vector: $\boldsymbol{x}^{(j)} = \text{argmin}_{\boldsymbol{x}^{(i)}}(\boldsymbol{x}^{(i)}\boldsymbol{w}^T)$.
5   **if** *all examples classified correctly* **then**
6   Return the solution $\boldsymbol{w}$. Terminate.
7   **else**
8   Update the weight vector: $\boldsymbol{w} = \boldsymbol{w} + \boldsymbol{x}^{(j)}$.
9   Go to 4.

[Ros62] Frank Rosenblatt. *Principles of Neurodynamics: Perceptron and the Theory of Brain Mechanisms*. Spartan Books, Washington, D.C., 1962.

Demo: Perceptron



Features of the perceptron algorithm

Perceptron convergence theorem [Nov62]:

- Perceptron algorithm eventually finds a hyperplane that separates 2 classes of points, if such a hyperplane exists.
- If no separating hyperplane exists, the alorithm does not have to converge and will iterate forever.

Possible solutions:

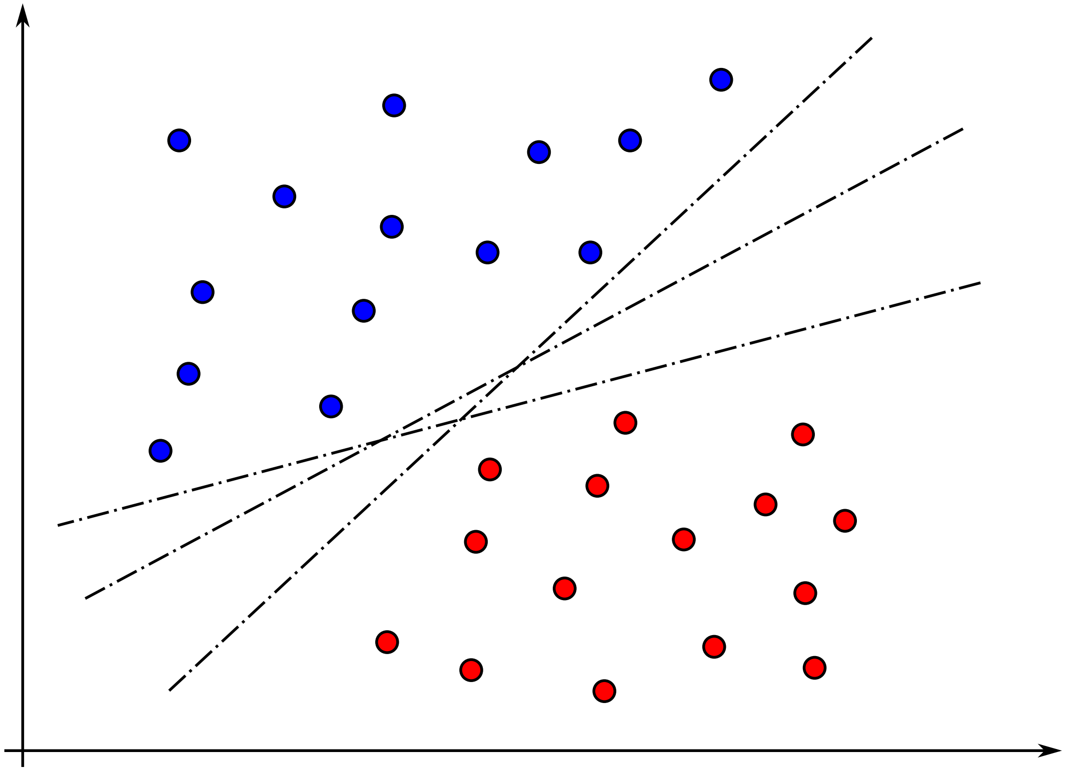
- Pocket algorithm - track the error the perceptron makes in each iteration and store the best weights found so far in a separate memory (pocket).
- Use a different learning algorithm, which finds an approximate solution, if the classes are not linearly separable.

[Nov62] Albert B. J. Novikoff. On convergence proofs for perceptrons. In *Proceedings of the Symposium on Mathematical Theory of Automata*, volume 12, Brooklyn, New York, 1962.

The hyperplane found by perceptron

The perceptron algorithm

- finds a separating hyperplane, if it exists;
- but if a single separating hyperplane exists, then there are infinitely many (equally good) separating hyperplanes



- and perceptron finds *any* of them!

Which separating hyperplane is the optimal one? What does "optimal" actually mean? (Possible answers in the SVM lecture.)

Logistic regression

Logistic regression model

Problem: Learn a binary **classifier** for the dataset $T = \{(\boldsymbol{x}^{(i)}, y^{(i)})\}$, where $y^{(i)} \in \{0, 1\}$.[1]

To reiterate: when using linear regression, the examples far from the decision boundary have a huge impact on h. How to limit their influence?

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Logistic regression uses a transformation of the values of linear function

$$h_{\boldsymbol{w}}(\boldsymbol{x}) = g(\boldsymbol{x}\boldsymbol{w}^T) = \frac{1}{1 + e^{-\boldsymbol{x}\boldsymbol{w}^T}},$$

where

$$g(z) = \frac{1}{1 + e^{-z}}$$

is the **sigmoid** function (a.k.a **logistic** function).

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Interpretation of the model:

- $h_{\boldsymbol{w}}(\boldsymbol{x})$ estimates the probability that $\boldsymbol{x}$ belongs to class 1.
- Logistic *regression* is a *classification model!*
- The discrimination function $h_{\boldsymbol{w}}(\boldsymbol{x})$ itself is not linear anymore; but the *decision boundary is still linear!*

[1] Previously, we have used $y^{(i)} \in \{-1, +1\}$, but the values can be chosen arbitrarily, and $\{0, 1\}$ is convenient for logistic regression.

Cost function

To train the logistic regression model, one can use the J_{MSE} criterion:

$$J(\boldsymbol{w}, T) = \frac{1}{|T|} \sum_{i=1}^{|T|} \left(y^{(i)} - h_{\boldsymbol{w}}(\boldsymbol{x}^{(i)}) \right)^2.$$

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Logistic regression uses a modified cost function

$$J(\boldsymbol{w}, T) = \frac{1}{|T|} \sum_{i=1}^{|T|} \mathrm{cost}(y^{(i)}, h_{\boldsymbol{w}}(\boldsymbol{x}^{(i)})), \text{ where}$$

$$\mathrm{cost}(y, \widehat{y}) = \begin{cases} -\log(\widehat{y}) & \text{if } y = 1 \\ -\log(1 - \widehat{y}) & \text{if } y = 0 \end{cases},$$

which can be rewritten in a single expression as

$$\mathrm{cost}(y, \widehat{y}) = -y \log(\widehat{y}) - (1 - y) \log(1 - \widehat{y}).$$

Such a cost function is simpler to optimize.