



# Electromagnetic Field Theory 1

*(fundamental relations and definitions)*

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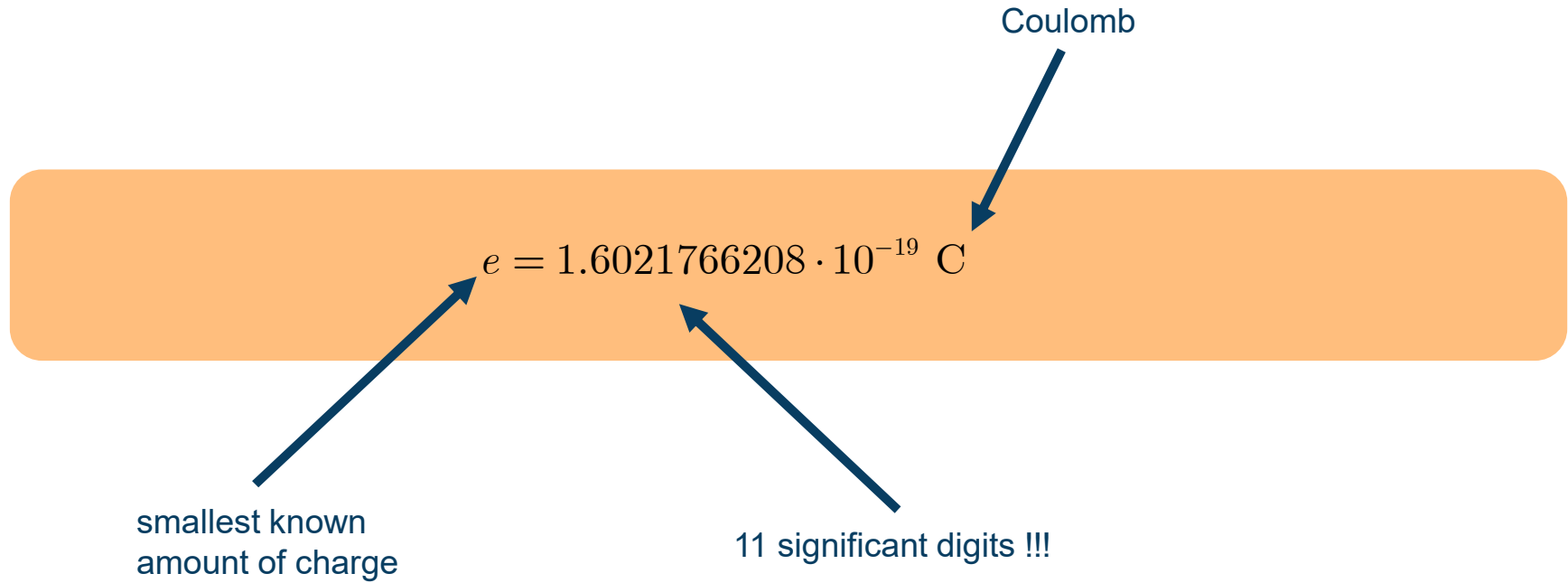


## Fundamental Question of Classical Electrodynamics

A specified distribution of elementary charges is in state of arbitrary (but known) motion. At certain time we pick one of them and ask what is the force acting on it.

*Rather difficult question – will not be fully answered*

# Elementary Charge



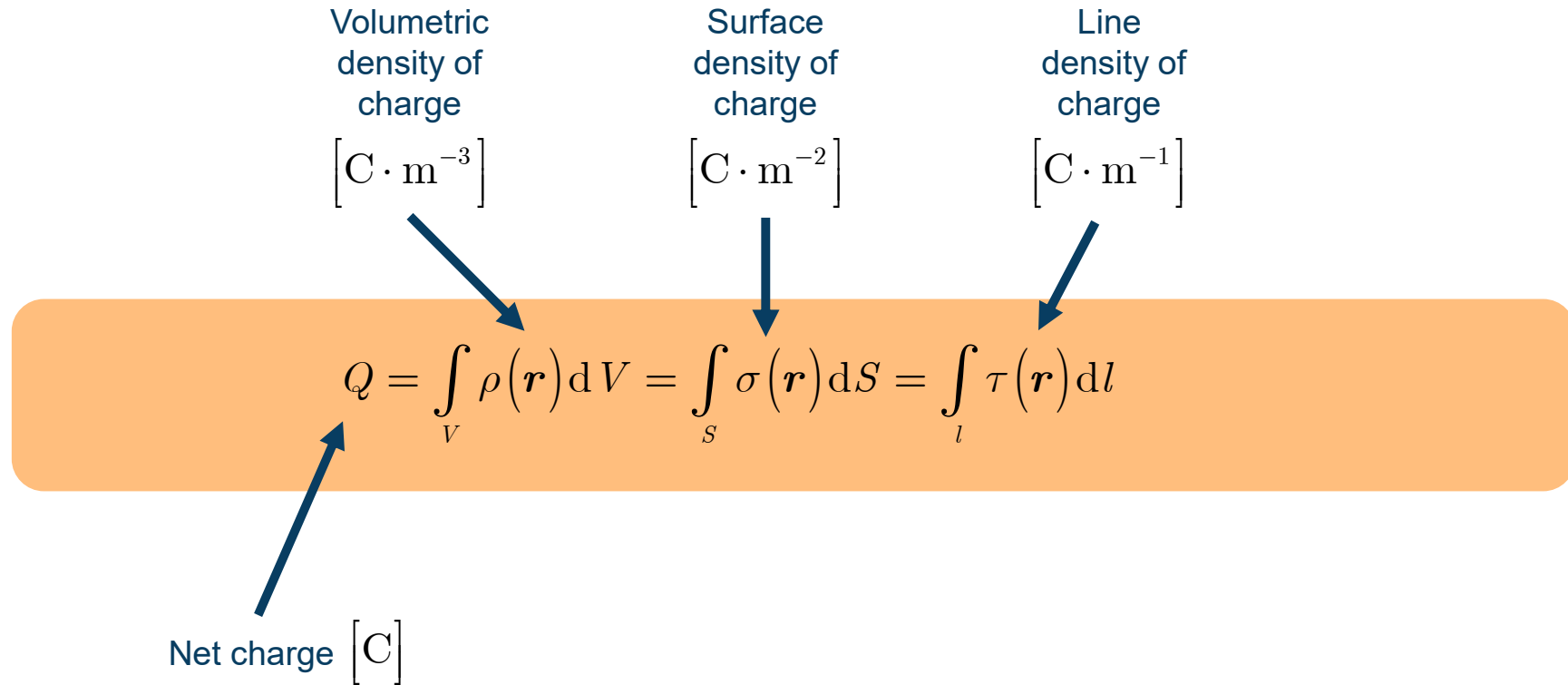
*As far as we known, all charges in nature have values  $\pm Ne, N \in \mathbb{Z}$*

## Charge conservation

Amount of charge is conserved in every frame (even non-inertial).

*Neutrality of atoms has been verified to 20 digits*

## Continuous approximation of charge distribution



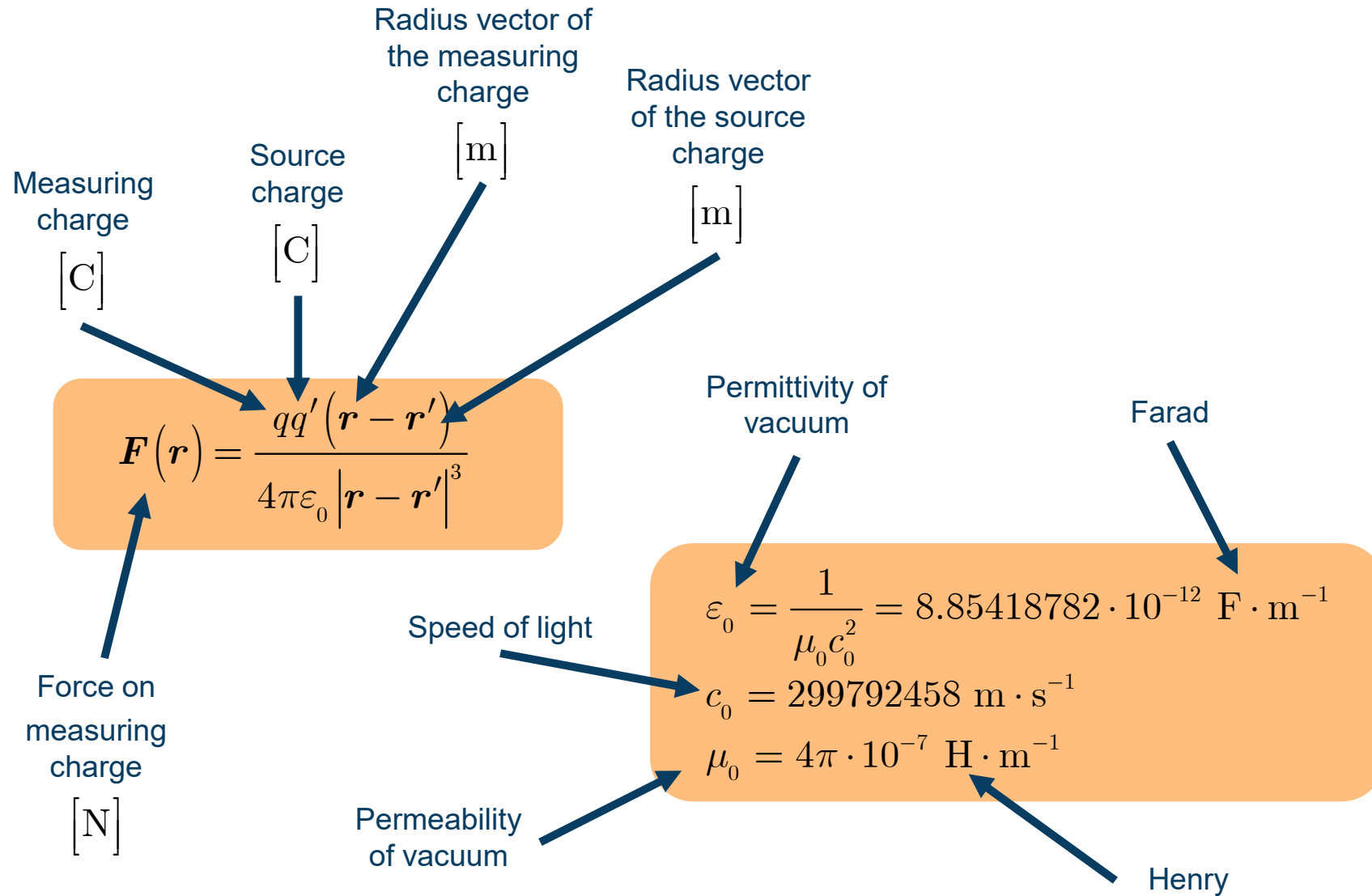
*Continuous approximation allows for using powerful mathematics*

## Fundamental Question of Electrostatics

There exist a specified distribution of static elementary charges. We pick one of them and ask what is the force acting on it.

*This will be answered in full details*

# Coulomb('s) Law



## Coulomb('s) Law + Superposition Principle

$$\mathbf{F}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \sum_n \frac{q'_n (\mathbf{r} - \mathbf{r}'_n)}{|\mathbf{r} - \mathbf{r}'_n|^3}$$

*Entire electrostatics can be deduced from this formula*



## Electric Field

$$\mathbf{F}(\mathbf{r}) = q\mathbf{E}(\mathbf{r}) \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_n \frac{q'_n (\mathbf{r} - \mathbf{r}'_n)}{|\mathbf{r} - \mathbf{r}'_n|^3}$$

Intensity of  
electric field  
 $[\text{V} \cdot \text{m}^{-1}]$

*Force is represented by field – entity generated by charges and permeating the space*

## Continuous Distribution of Charge

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_n \frac{q'_n (\mathbf{r} - \mathbf{r}'_n)}{|\mathbf{r} - \mathbf{r}'_n|^3} \quad \longrightarrow \quad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho(\mathbf{r}') (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV'$$

*Continuous description of charge allows for using powerful mathematics*

## Continuous Description of a Point Charge

Dirac's delta  
"function"



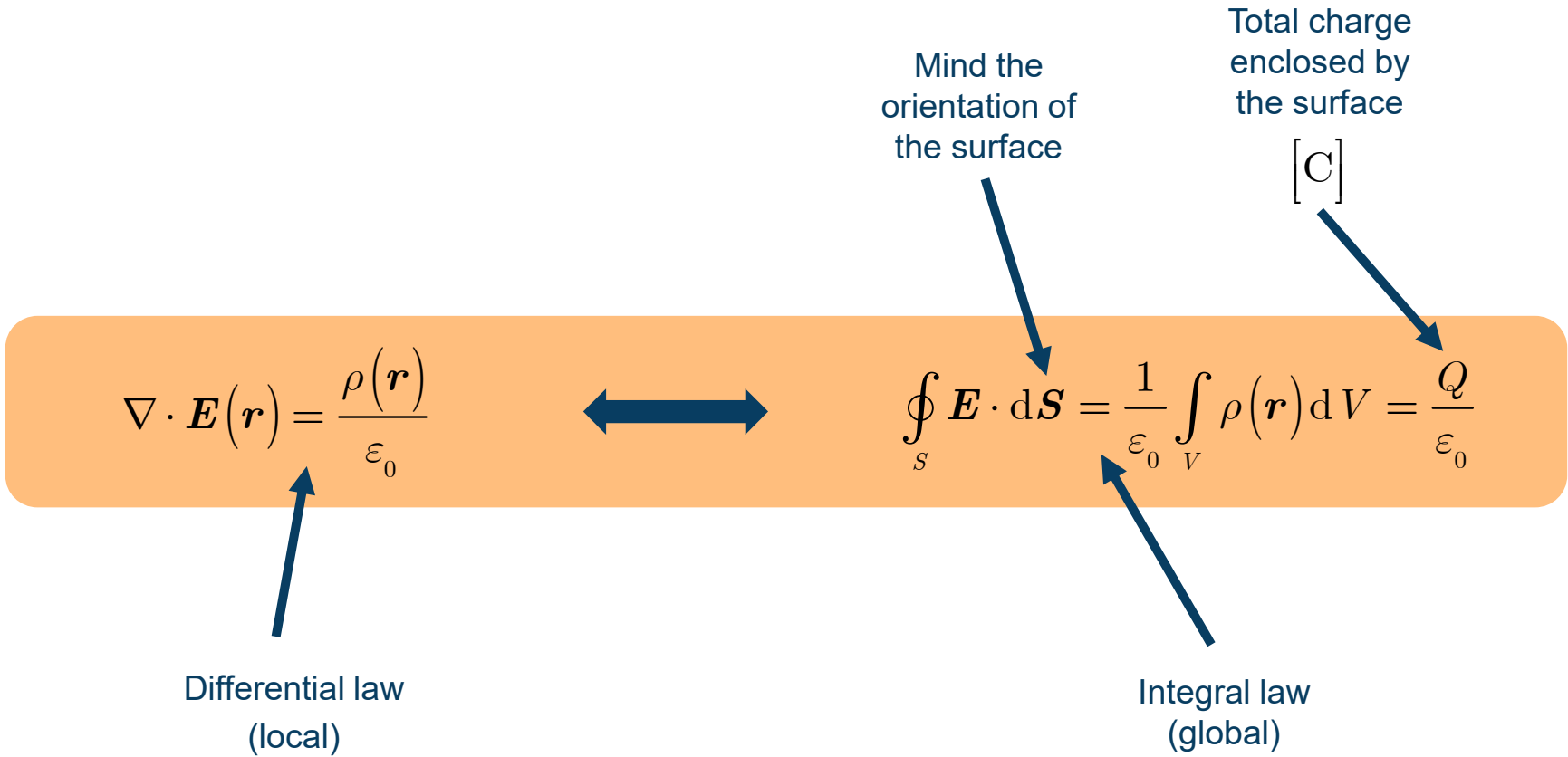
$$\rho(\mathbf{r}) = \sum_n q_n \delta(\mathbf{r} - \mathbf{r}_n)$$

Defining property of  
Dirac's delta "function"



$$\mathbf{F}(\mathbf{r}_n) = \int_V \mathbf{F}(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}_n) dV$$

# Gauss(') Law



## Rotation of Electric Field



## Various Views on Electrostatics

Integral laws of electrostatics



Differential laws of electrostatics



Coulomb's law



$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0}$$

$$\oint_l \mathbf{E} \cdot d\mathbf{l} = 0$$



$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0}$$

$$\nabla \times \mathbf{E} = 0$$



$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV'$$

*The physics content is the same, the formalism is different.*

# Electric potential

Electric potential

Defined up to arbitrary constant

$$\nabla \times \mathbf{E} = 0 \quad \Rightarrow \quad \mathbf{E}(\mathbf{r}) = -\nabla\varphi(\mathbf{r}) \quad \Rightarrow \quad \varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' + K$$

*Scalar description of electrostatic field*

# Voltage

Potential difference is a unique number

Voltage  
[V]

Work necessary to take charge  $q$  from point A to point B

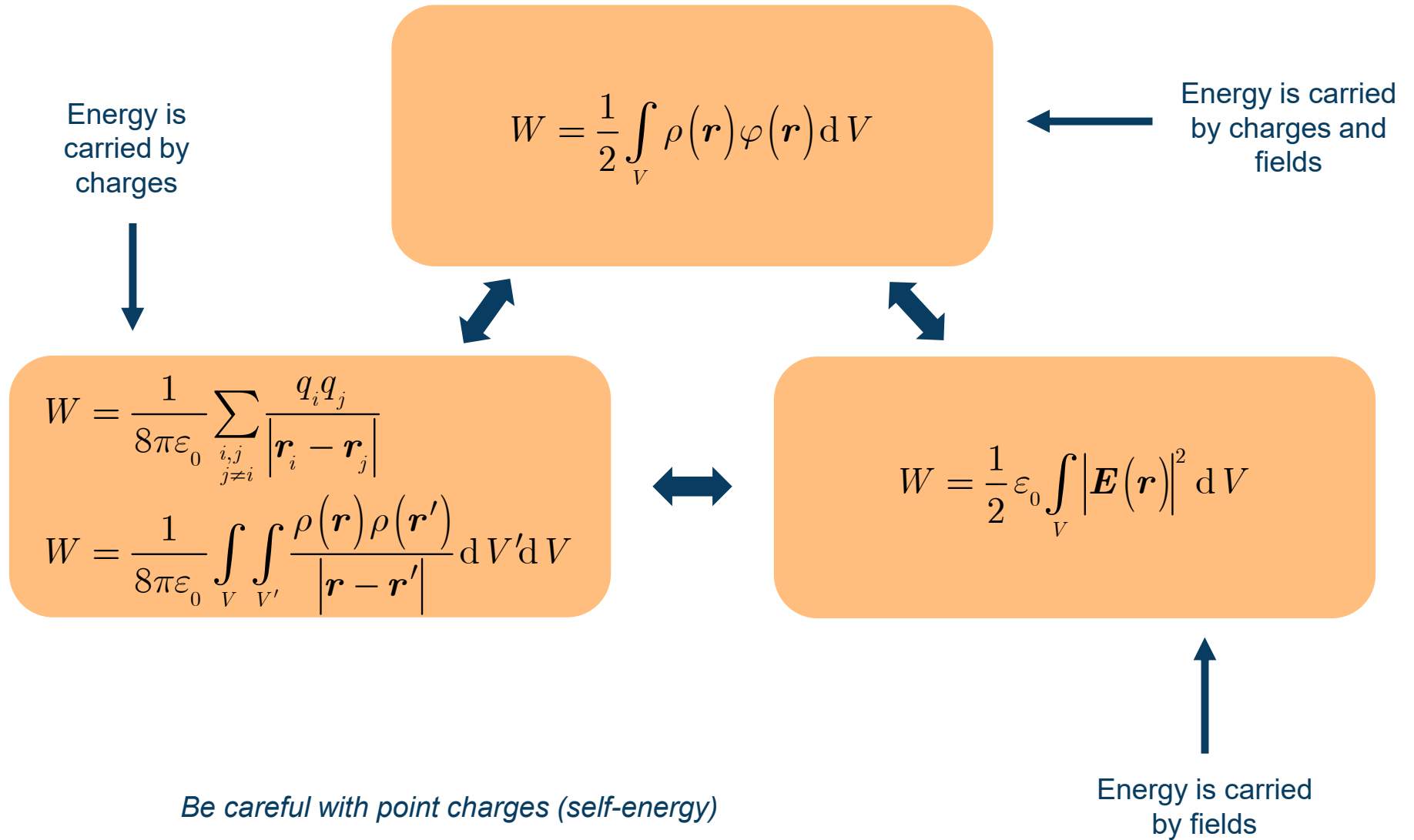
$$-\int_A^B \mathbf{E} \cdot d\mathbf{l} = \varphi(B) - \varphi(A) = U$$

$$W = -\int_A^B \mathbf{F} \cdot d\mathbf{l} = qU$$

*Voltage represents connection of abstract field theory with experiments*



# Electrostatic Energy



## Electrostatic Energy vs Force

Energy of a system of point charges



Coulomb's law



$$W = \frac{1}{8\pi\epsilon_0} \sum_{\substack{i,j \\ j \neq i}} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} \quad \Rightarrow \quad \mathbf{F}(\mathbf{r}_\xi) = -\nabla_\xi W = \frac{q_\xi}{4\pi\epsilon_0} \sum_{\substack{j \\ j \neq \xi}} q_j \frac{(\mathbf{r}_\xi - \mathbf{r}_j)}{|\mathbf{r}_\xi - \mathbf{r}_j|^3}$$

*Electrostatic forces are always acting so to minimize energy of the system*

## Conductors

Ideal conductor contains unlimited amount of free charges which under action of external electric field rearrange so as to annihilate electric field inside the conductor.

In 3D, the free charge always resides on the external bounding surface of the conductor.



In 1D and 2D  
it is not so

*Generally free charges in conductors move so as to minimize the energy*

## Boundary Conditions on Conductors

- Inside conductor

- $\mathbf{E}(\mathbf{r}) = 0 \Leftrightarrow \varphi(\mathbf{r}) = \text{const.}$

Potential is continuous across the boundary

- Just outside conductor

- $\mathbf{n}(\mathbf{r}) \times \mathbf{E}(\mathbf{r}) = 0 \Leftrightarrow \varphi(\mathbf{r}) = \text{const.}$

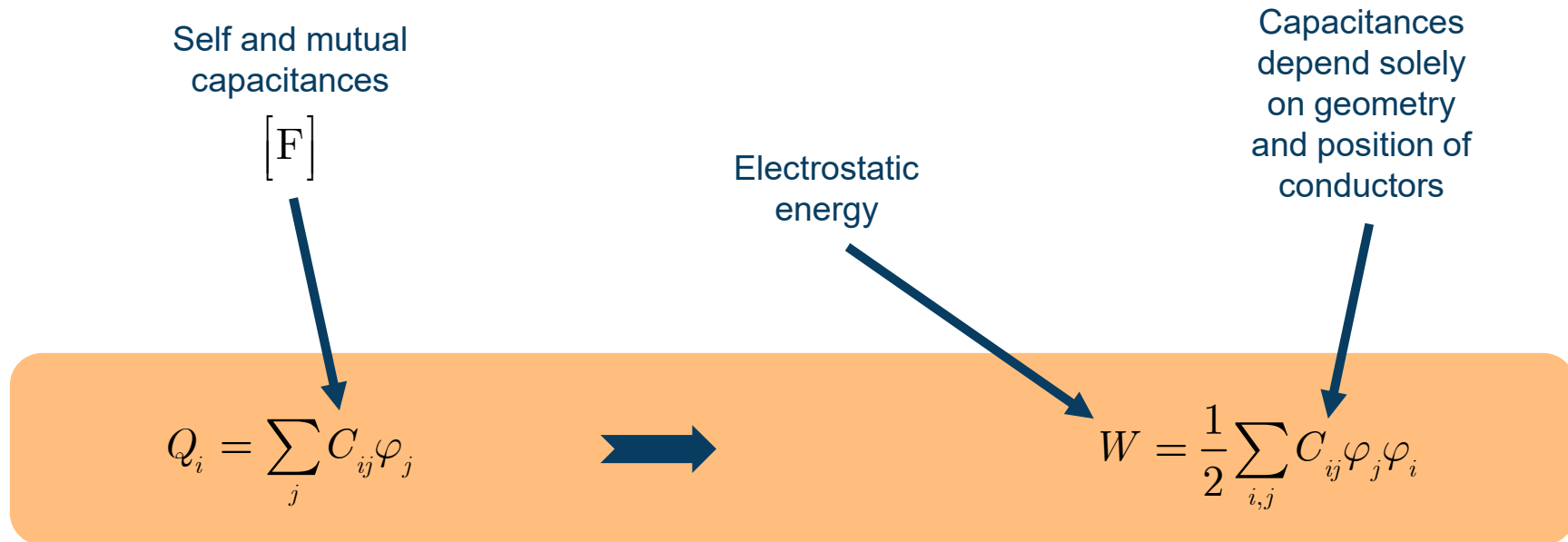
Surface charge residing on the outer surface of the conductor

- $\mathbf{n}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) = \frac{\sigma}{\epsilon_0} \Leftrightarrow \frac{\partial \varphi(\mathbf{r})}{\partial n} = -\frac{\sigma}{\epsilon_0}$

Outward normal to the conductor

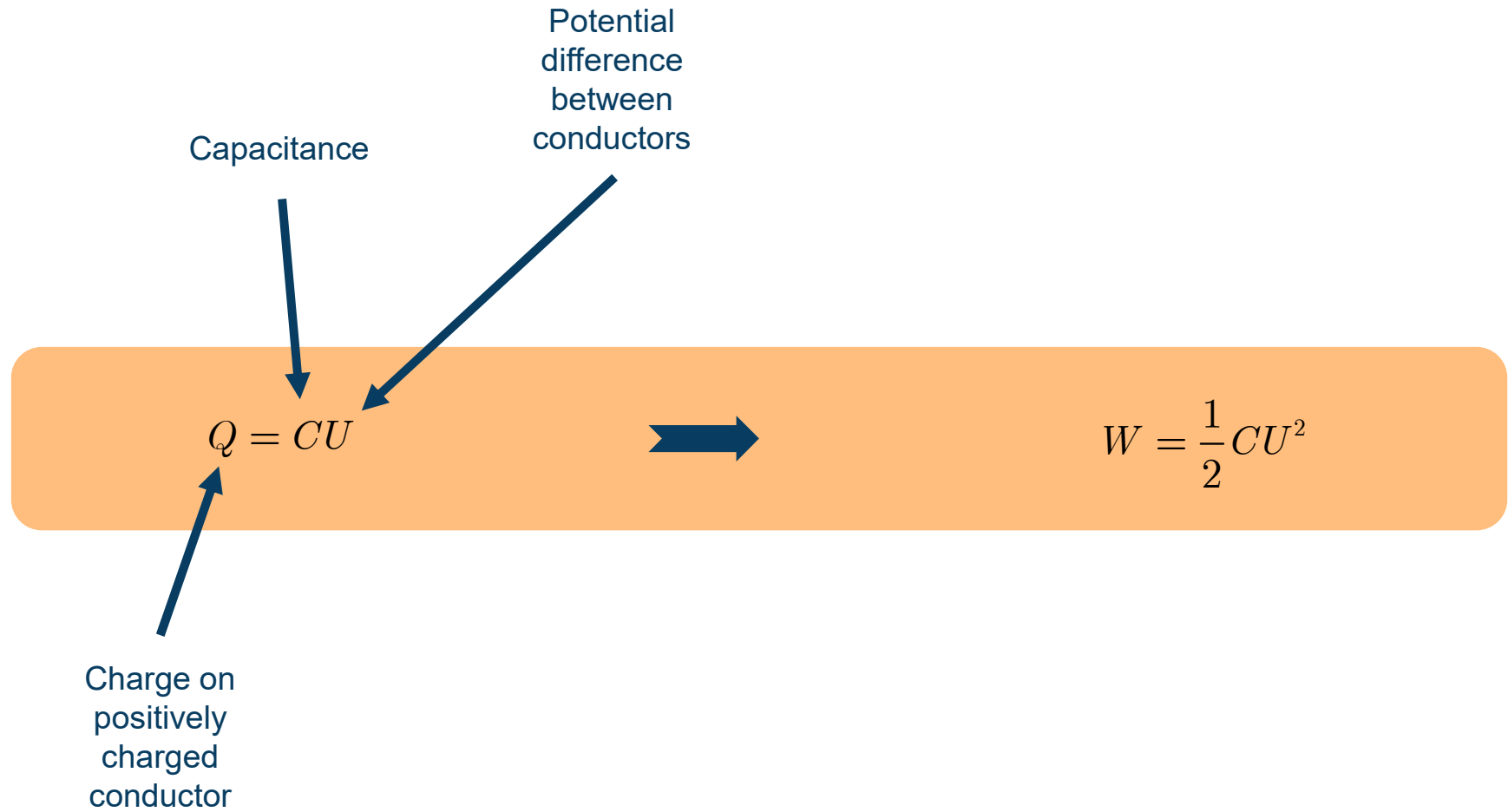
Normal derivative

## Capacitance of a System of $N$ conductors



*Electrostatic system is fully characterized by capacitances (we know the energy)*

# Capacitance of a System of two conductors



## Poisson('s) equation

$$\Delta\varphi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

*The solution to Poisson's equation is unique in a given volume once the potential is known on its bounding surface and the charge density is known through out the volume.*

## Laplace('s) equation

$$\Delta\varphi(\mathbf{r}) = 0$$

*The solution to Laplace's equation is unique in a given volume once the potential is known on its bounding surface.*



## Mean Value Theorem

Center of the sphere

Only for spheres containing no charge

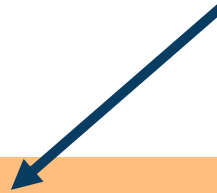
$$\varphi(\mathbf{r}_{\text{center}}) = \frac{1}{4\pi R^2} \oint_{\text{sphere}} \varphi(\mathbf{r}) dS$$

Radius of the sphere

*The solution to Laplace's equation possesses neither maxima nor minima inside the solved volume.*

## Earnshaw('s) Theorem

Consequence of  
mean value theorem



A charged particle cannot be held in stable equilibrium by electrostatic forces alone.

*Mind that the solution to Laplace's equation posses neither maxima nor minima inside the solved volume. This means that charged particle will always travel towards the boundary.*

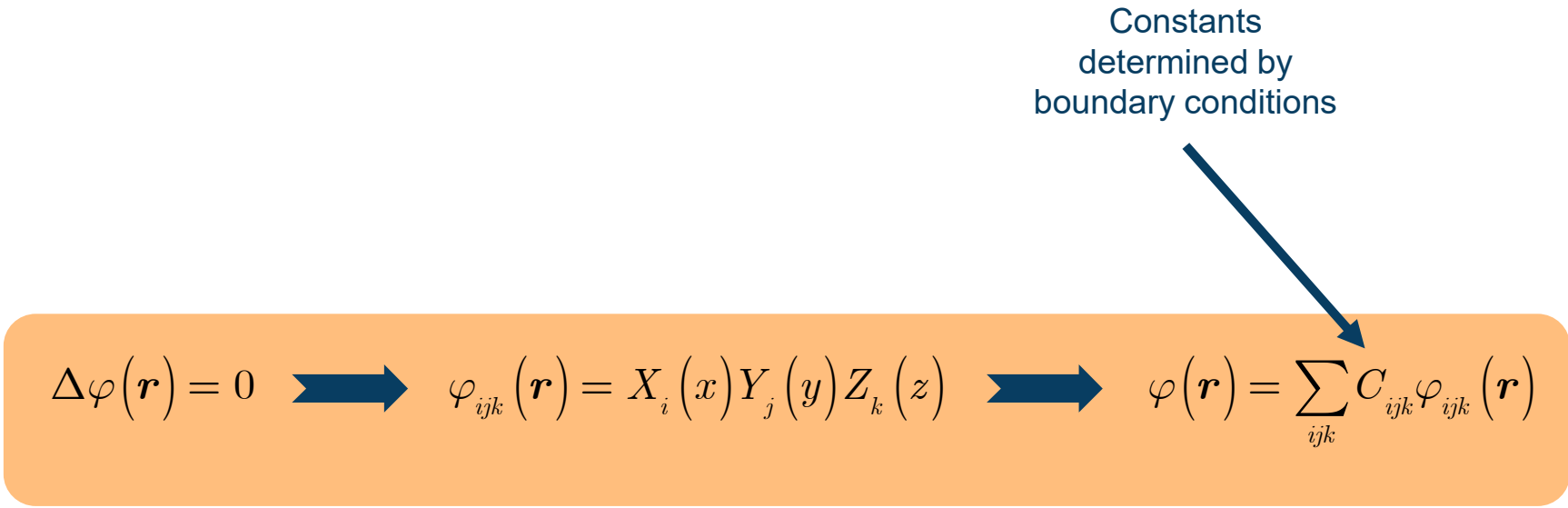
## Image Method

When solving field generated by charges in the presence of conductors, it is sometimes possible to remove the conductor and mimic its boundary conditions by adding extra charges to the exterior of the solution volume. The unicity theorem claims that this is a correct solution.

*Image method always works with planes and spheres.*

## Separation of Variables

Constants  
determined by  
boundary conditions


$$\Delta\varphi(\mathbf{r}) = 0 \quad \Rightarrow \quad \varphi_{ijk}(\mathbf{r}) = X_i(x)Y_j(y)Z_k(z) \quad \Rightarrow \quad \varphi(\mathbf{r}) = \sum_{ijk} C_{ijk} \varphi_{ijk}(\mathbf{r})$$

*Semi-analytical method for canonical problems*

## Finite Differences

$$\varphi(x+h, y, z) \rightarrow \varphi_{(i+1)jk}$$

$$\Delta\varphi(\mathbf{r}) \approx \frac{\varphi_{(i+1)jk} - 2\varphi_{ijk} + \varphi_{(i-1)jk}}{h^2} + \frac{\varphi_{i(j+1)k} - 2\varphi_{ijk} + \varphi_{i(j-1)k}}{h^2} + \frac{\varphi_{ij(k+1)} - 2\varphi_{ijk} + \varphi_{ij(k-1)}}{h^2}$$

$$\Delta\varphi(\mathbf{r}) = 0 \quad \Rightarrow \quad \varphi_{ijk} = \frac{\varphi_{(i+1)jk} + \varphi_{(i-1)jk} + \varphi_{i(j+1)k} + \varphi_{i(j-1)k} + \varphi_{ij(k+1)} + \varphi_{ij(k-1)}}{6}$$

Approximation by a system of linear algebraic equations

Mind the mean value theorem

*Powerful numerical method for closed problems*

# Method of Moments

Assumed to be known in volume where the charge resides

Distribution of charge is unknown

Simple functions for which the potential integral can be easily evaluated

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'$$

$$\rho(\mathbf{r}) \approx \sum_n \alpha_n \rho_n(\mathbf{r})$$

$$\int_V \rho_m(\mathbf{r}) \varphi(\mathbf{r}) dV = \sum_n \alpha_n \frac{1}{4\pi\epsilon_0} \int_V \int_{V'} \frac{\rho_m(\mathbf{r}) \rho_n(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' dV$$

Known

Approximation by a system of linear algebraic equations

Known

*Powerful numerical method for open problems*

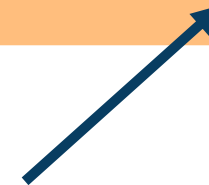
## Dielectrics

- Material in which **charges cannot move freely**
- Charges are forming **clusters (atoms, molecules)**
- Under influence of electric field the clusters **change shape or rotate**
- Electric field induces **electric dipoles** with density  $P(r)$   $[C \cdot m^{-2}]$

Clusters are electrically neutral



Number of dipoles in unitary volume



## Electric Field of a Dipole

Two opposite charges  
very close to each other

$$|\mathbf{r} - \mathbf{r}_{\text{center}}| \gg |\mathbf{r}_1 - \mathbf{r}_2|$$

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\mathbf{r} - \mathbf{r}_1|} - \frac{1}{4\pi\epsilon_0} \frac{q}{|\mathbf{r} - \mathbf{r}_2|} \approx \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}_{\text{center}})}{|\mathbf{r} - \mathbf{r}_{\text{center}}|^3}$$

$$\mathbf{p} = q(\mathbf{r}_1 - \mathbf{r}_2) = \int_V \mathbf{r} \rho(\mathbf{r}) dV$$

Electric dipole  
moment  
[C · m]

Formula for two  
opposite charges

General formula



## Field Produced by Polarized Matter

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV' = \frac{1}{4\pi\epsilon_0} \oint_{S'} \frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \cdot d\mathbf{S}' - \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'$$

Only apply at infinitely sharp boundary (unrealistic)

Potential of volumetric charge density

*This formula holds very well outside the matter and, curiously, it also well approximates the field inside*

# Electric Displacement

Electric displacement  $[\text{C} \cdot \text{m}^{-2}]$

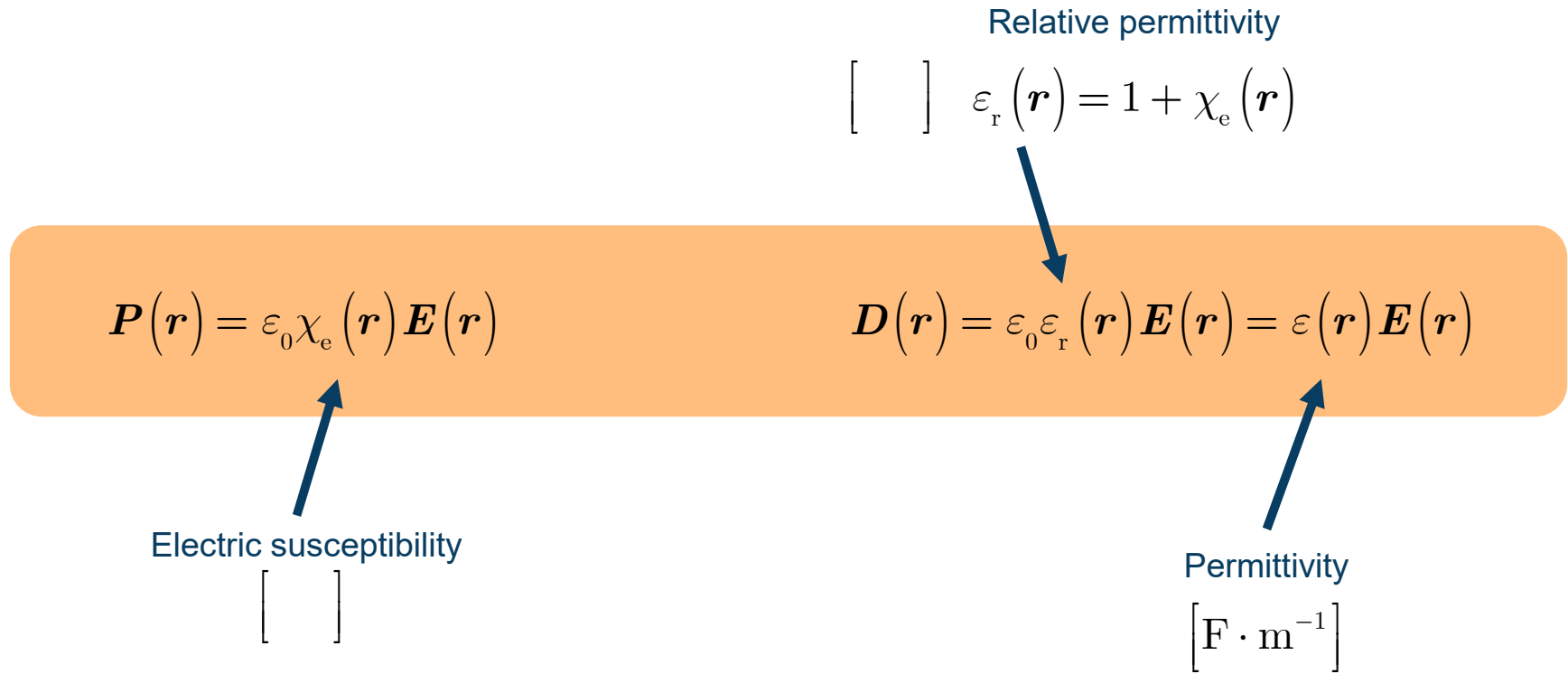
$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho(\mathbf{r})$$

$$\mathbf{D}(\mathbf{r}) = \epsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho(\mathbf{r}) dV = Q$$

Only free charge  
(compare to divergence  
of electric field)

# Linear Isotropic Dielectrics




*All the complicated structure of matter reduces to a simple scalar quantity*

## Fields in Presence of Dielectrics 1/2

Analogy with electric field in vacuum can only be used when entire space is homogeneously filled with dielectric.

$$\nabla \times \mathbf{D}(\mathbf{r}) = \nabla \times [\varepsilon(\mathbf{r}) \mathbf{E}(\mathbf{r})] \neq 0$$



Inequality is due to boundaries

*Analogy with vacuum can only be used when space is homogeneously filled with dielectric*

## Fields in Presence of Dielectrics 2/2

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0 \Leftrightarrow \mathbf{E}(\mathbf{r}) = -\nabla\varphi(\mathbf{r}) \quad \Rightarrow \quad \nabla \cdot [\varepsilon(\mathbf{r})\nabla\varphi(\mathbf{r})] = -\rho(\mathbf{r})$$

$$\Delta\varphi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon}$$

Not a function of  
coordinates

*Poisson's equation holds only when permittivity does not depend on coordinates*

## Dielectric Boundaries

$$\mathbf{n}(\mathbf{r}) \times [\mathbf{E}_1(\mathbf{r}) - \mathbf{E}_2(\mathbf{r})] = 0 \quad \Leftrightarrow \quad \varphi_1(\mathbf{r}) - \varphi_2(\mathbf{r}) = 0$$

$$\mathbf{n}(\mathbf{r}) \cdot [\varepsilon_1 \mathbf{E}_1(\mathbf{r}) - \varepsilon_2 \mathbf{E}_2(\mathbf{r})] = \sigma \quad \Leftrightarrow \quad \varepsilon_1 \frac{\partial \varphi_1(\mathbf{r})}{\partial n} - \varepsilon_2 \frac{\partial \varphi_2(\mathbf{r})}{\partial n} = -\sigma$$

Normal  
pointing to  
region (1)

*Both conditions are needed for unique solution*

## Electrostatic Energy in Dielectrics

$$W = \frac{1}{2} \epsilon_0 \int_V |\mathbf{E}(\mathbf{r})|^2 dV$$



$$W = \frac{1}{2} \int_V \mathbf{E}(\mathbf{r}) \cdot \mathbf{D}(\mathbf{r}) dV$$

## Forces on Dielectrics

$$W = \frac{1}{2} C U^2 = \frac{1}{2} \frac{Q^2}{C}$$

$$W = \frac{1}{2} \int_V \mathbf{E}(\mathbf{r}) \cdot \mathbf{D}(\mathbf{r}) dV$$



$$\mathbf{F}(\mathbf{r}_\xi) = -\nabla_\xi W$$

This only holds when charge is held constant



# Electric Current

Current density  
 $[A \cdot m^{-2}]$

Charge  
 $[C]$

Velocity of charge  
 $[m \cdot s^{-1}]$

$$\mathbf{J}(\mathbf{r}, t) = \sum_{k=1}^N q_k \delta(\mathbf{r} - \mathbf{r}_k(t)) \mathbf{v}_k(t)$$

Volumetric density  
represented by  
Dirac delta

$$[m^{-3}]$$

*Charges in motion are represented by current density*

## Local Charge Conservation

$$\nabla \cdot \mathbf{J}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \sum_{k=1}^N q_k \delta(\mathbf{r} - \mathbf{r}_k(t)) = -\frac{\partial \rho(\mathbf{r}, t)}{\partial t}$$

*Charge is conserved locally at every space-time point*

## Global Charge Conservation

When charge leaves a given volume, it is always accompanied by a current through the bounding envelope



$$\oint_s \mathbf{J}(\mathbf{r}, t) \cdot d\mathbf{S} = -\frac{\partial Q(t)}{\partial t}$$

*Charge can neither be created nor destroyed. It can only be displaced.*

## Stationary Current

When charge enters a volume, another must leave it  
without any delay



$$\nabla \cdot \mathbf{J}(\mathbf{r}) = 0$$



$$\oint_S \mathbf{J}(\mathbf{r}) \cdot d\mathbf{S} = 0$$

*There is no charge accumulation in stationary flow*

## Electromotive Force

Stationary flow of charges cannot be caused by electrostatic field. The motion forces are non-conservative, are called electromotive forces, and are commonly of chemical, magnetic or photoelectric origin.

$$\oint_l \mathbf{E}(\mathbf{r}) \cdot d\mathbf{l} \neq 0$$

For curves passing through sources of electromotive force

$$\oint_l \mathbf{E}(\mathbf{r}) \cdot d\mathbf{l} = 0$$

For curves not crossing sources of electromotive force

## Ohm('s) Law

Conductivity

$$[\text{S} \cdot \text{m}^{-1}]$$



$$\mathbf{J}(\mathbf{r}) = \sigma(\mathbf{r}) \mathbf{E}(\mathbf{r})$$

*This simple linear relation holds for enormous interval of electric field strengths*

## Boundary Conditions for Stationary Current

$$\mathbf{n}(\mathbf{r}) \times [\mathbf{E}_1(\mathbf{r}) - \mathbf{E}_2(\mathbf{r})] = 0 \quad \Leftrightarrow \quad \varphi_1(\mathbf{r}) - \varphi_2(\mathbf{r}) = 0$$

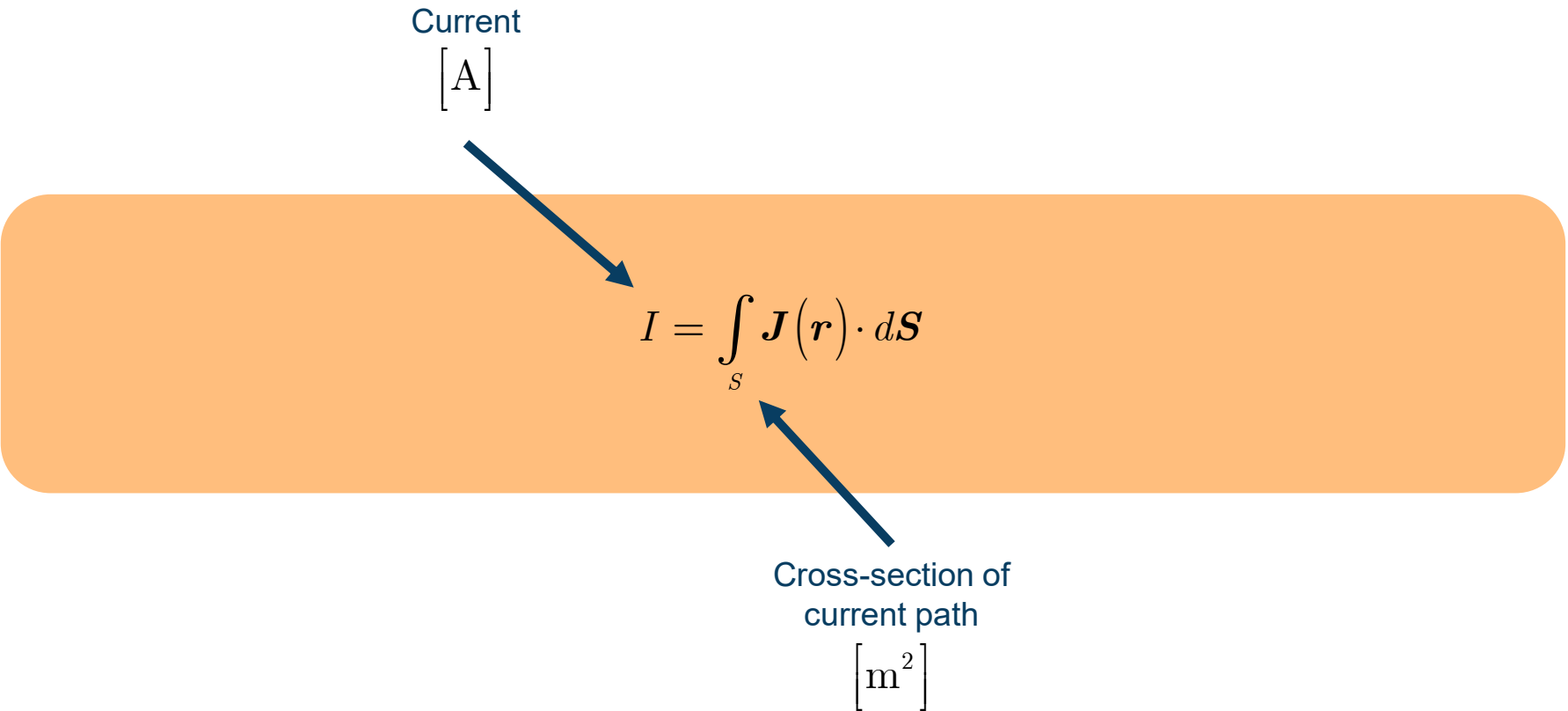
$$\mathbf{n}(\mathbf{r}) \cdot [\varepsilon_1 \mathbf{E}_1(\mathbf{r}) - \varepsilon_2 \mathbf{E}_2(\mathbf{r})] = \sigma \quad \Leftrightarrow \quad \varepsilon_1 \frac{\partial \varphi_1(\mathbf{r})}{\partial n} - \varepsilon_2 \frac{\partial \varphi_2(\mathbf{r})}{\partial n} = -\sigma$$

$$\mathbf{n}(\mathbf{r}) \cdot [\sigma_1 \mathbf{E}_1(\mathbf{r}) - \sigma_2 \mathbf{E}_2(\mathbf{r})] = 0 \quad \Leftrightarrow \quad \sigma_1 \frac{\partial \varphi_1(\mathbf{r})}{\partial n} - \sigma_2 \frac{\partial \varphi_2(\mathbf{r})}{\partial n} = 0$$

*Charge conservation forces the continuity of current across the boundary*

# Electric Current

Current  
[A]

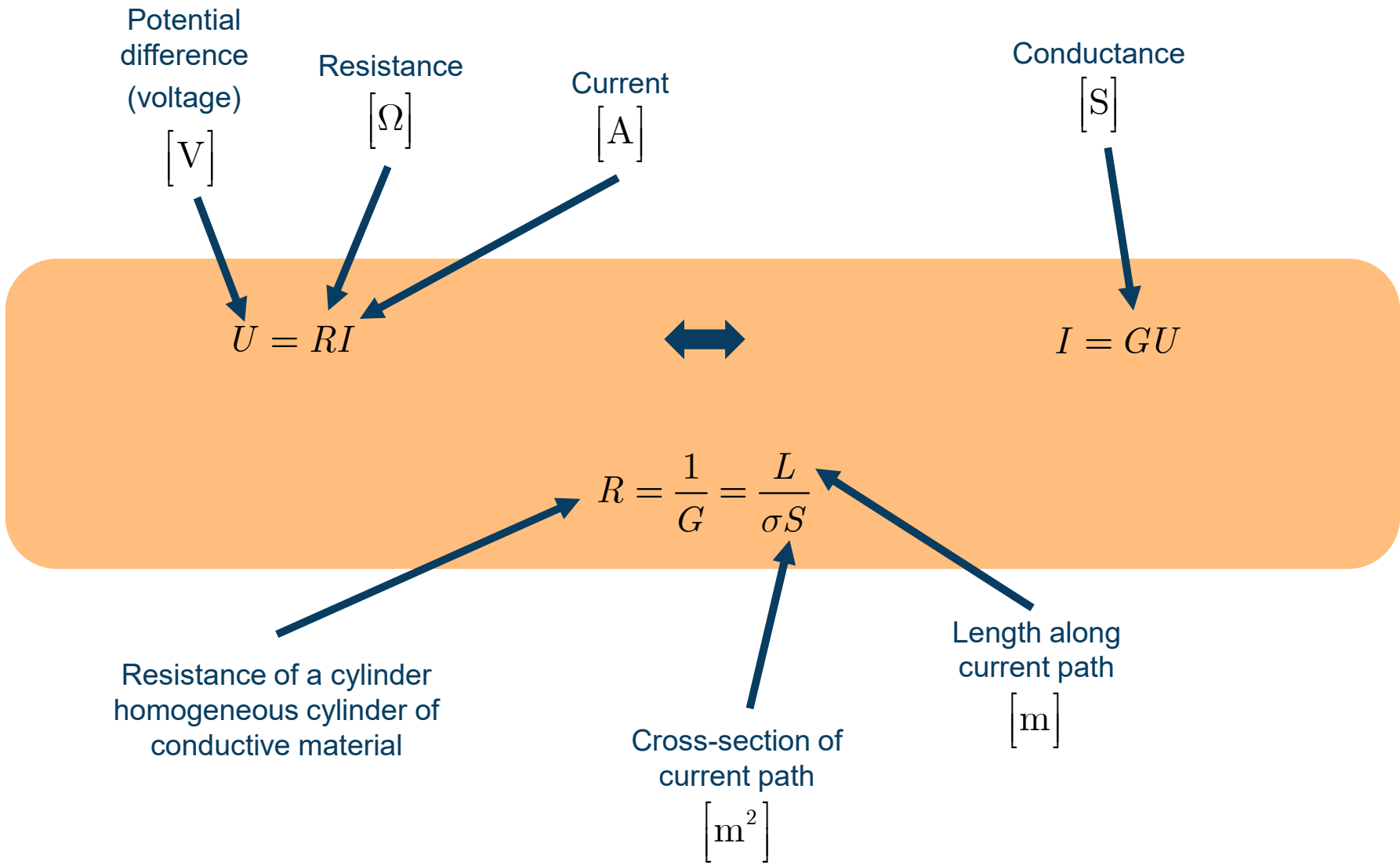

$$I = \int_S \mathbf{J}(\mathbf{r}) \cdot d\mathbf{S}$$

Cross-section of  
current path  
[m<sup>2</sup>]

*Existence of high contrast in conductivity between conductors and dielectrics allows for well defined current paths.*



# Resistance (Conductance)



## Resistive Circuits and Kirchhoff('s) Laws

In a loop

$$\sum_i U_i = U_{\text{electromotive}}$$

On a resistor

$$U_i = R_i I_i$$

At a junction

$$\sum_i I_i = 0$$

*Kirchhoff's laws are a consequence of electrostatics and law's of stationary current flow*

## Joule('s) Heat

Power lost via  
conduction

[W]

Power lost on  
resistor

[W]

$$P = \int_V \mathbf{E}(\mathbf{r}) \cdot \mathbf{J}(\mathbf{r}) dV = \int_V \sigma(\mathbf{r}) |\mathbf{E}(\mathbf{r})|^2 dV$$



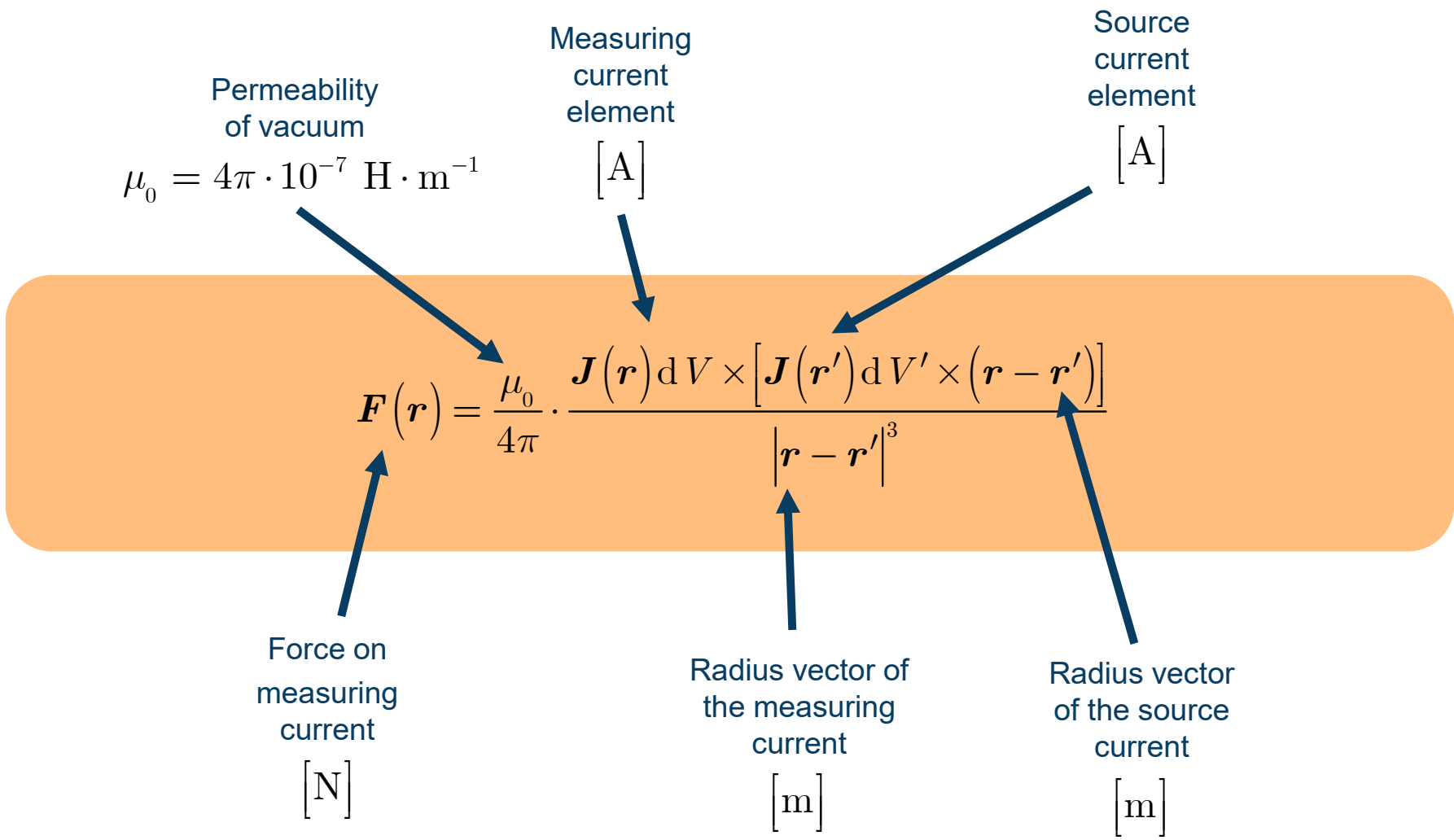
$$P = UI = RI^2 = \frac{U^2}{R}$$

*Electric field within conducting material produce heat*

## Fundamental Question of Magnetostatics

There exist a specified distribution of stationary current. We pick a differential volume of it and ask what is the force acting on it.

# Biot-Savart('s) Law



## Biot-Savart('s) Law + Superposition Principle

$$\mathbf{F}(\mathbf{r}) = \mathbf{J}(\mathbf{r}) dV \times \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV'$$

*Entire magnetostatics can be deduced from this formula*

## Magnetic Field

$$\mathbf{F}(\mathbf{r}) = \mathbf{J}(\mathbf{r}) dV \times \mathbf{B}(\mathbf{r})$$



$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV'$$

Magnetic field  
(Magnetic induction)  
[T]

## Divergence of Magnetic Field

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$



$$\oint_S \mathbf{B}(\mathbf{r}) \cdot d\mathbf{S} = 0$$

*There are no point sources of magnetostatic field*



## Curl of Magnetic Field – Ampere('s) Law

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r})$$



$$\oint_l \mathbf{B}(\mathbf{r}) \cdot d\mathbf{l} = \mu_0 I$$

Total current  
captured within  
the curve

[A]

# Magnetic Vector Potential

Magnetic vector potential

Defined up to arbitrary scalar function

$$\nabla \cdot \mathbf{B} = 0 \implies \mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) \implies \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' + \nabla \psi(\mathbf{r})$$

*Reduced description of magnetostatic field*

## Poisson('s) equation

$$\Delta \mathbf{A}(\mathbf{r}) = -\mu_0 \mathbf{J}(\mathbf{r})$$

*The solution to Poisson's equation is unique in a given volume once the potential is known on its bounding surface and the current density is known through out the volume.*

## Boundary Conditions

Surface current on the boundary

$$\mathbf{n}(\mathbf{r}) \times [\mathbf{B}_1(\mathbf{r}) - \mathbf{B}_2(\mathbf{r})] = \mu_0 \mathbf{K}(\mathbf{r})$$
$$\mathbf{n}(\mathbf{r}) \cdot [\mathbf{B}_1(\mathbf{r}) - \mathbf{B}_2(\mathbf{r})] = 0$$
$$\mathbf{A}_1(\mathbf{r}) - \mathbf{A}_2(\mathbf{r}) = 0$$

Normal pointing to region (1)

## Magnetostatic Energy

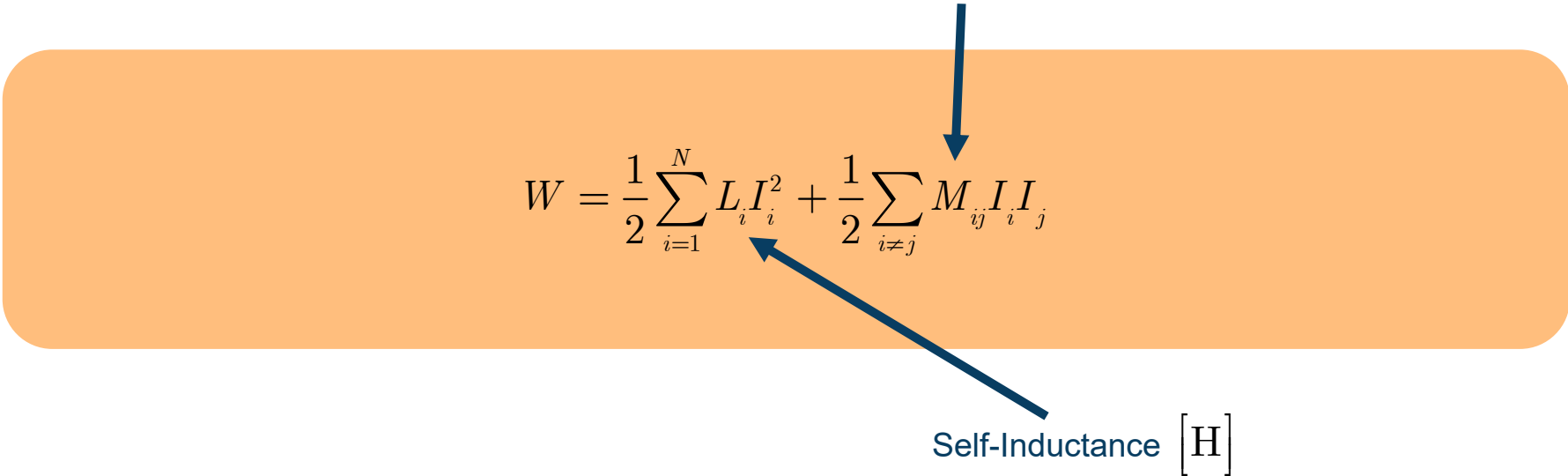
$$W = \frac{1}{2} \int_V \mathbf{A}(\mathbf{r}) \cdot \mathbf{J}(\mathbf{r}) dV \quad \longleftrightarrow \quad W = \frac{1}{2\mu_0} \int_V |\mathbf{B}(\mathbf{r})|^2 dV$$

*For now it is just a formula that works – it must be derived with the help of time varying fields*

## Magnetostatic Energy – Current Circuits

$$M_{ij} = M_{ji} = \frac{\mu_0}{4\pi I_i I_j} \int_{V_j} \int_{V'_i} \frac{\mathbf{J}_j(\mathbf{r}_j) \cdot \mathbf{J}_i(\mathbf{r}'_i)}{|\mathbf{r}_j - \mathbf{r}'_i|} dV'_i dV_j$$

Mutual-Inductance [H]


$$W = \frac{1}{2} \sum_{i=1}^N L_i I_i^2 + \frac{1}{2} \sum_{i \neq j} M_{ij} I_i I_j$$

Self-Inductance [H]

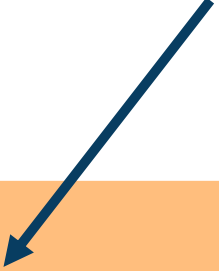
$$L_i = \frac{\mu_0}{4\pi I_i^2} \int_{V_i} \int_{V'_i} \frac{\mathbf{J}_i(\mathbf{r}_i) \cdot \mathbf{J}_i(\mathbf{r}'_i)}{|\mathbf{r}_i - \mathbf{r}'_i|} dV'_i dV_i$$

## Mutual Inductance – Thin Current Loop

$$\Phi_{ji} = \int_{S_j} \mathbf{B}_i(\mathbf{r}_j) \cdot d\mathbf{S}_j$$

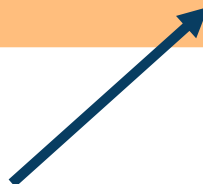
Magnetic flux induced by  $i$ -th current through  $j$ -th current

[Wb]


$$M_{ij} = \frac{\Phi_{ji}}{I_i}$$

## Magnetic Materials

- Material response is due to magnetic dipole moments
- Magnetic moment comes from spin or orbital motion of an electron
- Magnetic field tends to align magnetic moments
- Magnetic field induces magnetic dipoles with density  $M(\mathbf{r})$   $[\text{A} \cdot \text{m}^{-1}]$



Number of dipoles  
in unitary volume



## Magnetic Field of a Dipole

Dipole is assumed at the origin

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \cdot \frac{\mathbf{m} \times \mathbf{r}}{r^3}$$



$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \left[ \frac{3\mathbf{r}(\mathbf{r} \cdot \mathbf{m})}{r^5} - \frac{\mathbf{m}}{r^3} \right]$$

$\mathbf{r} \neq 0$

$$\mathbf{m} = \frac{1}{2} \int_V \mathbf{r} \times \mathbf{J}(\mathbf{r}) dV$$

Magnetic dipole moment

$$[\text{A} \cdot \text{m}^2]$$

*Magnetic dipole approximates infinitesimally small current loop*

## Field Produced by Magnetized Matter

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{M}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV' = \frac{\mu_0}{4\pi} \oint_{S'} \frac{\mathbf{M}(\mathbf{r}') \times d\mathbf{S}'}{|\mathbf{r} - \mathbf{r}'|} + \frac{\mu_0}{4\pi} \int_{V'} \frac{\nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'$$

Only applies at infinitely sharp boundary (unrealistic)

Potential of volumetric current density

*This formula holds very well outside the matter and, curiously, it also well approximates the field inside*

# Magnetic Intensity

Magnetic Intensity  $[\text{A} \cdot \text{m}^{-1}]$

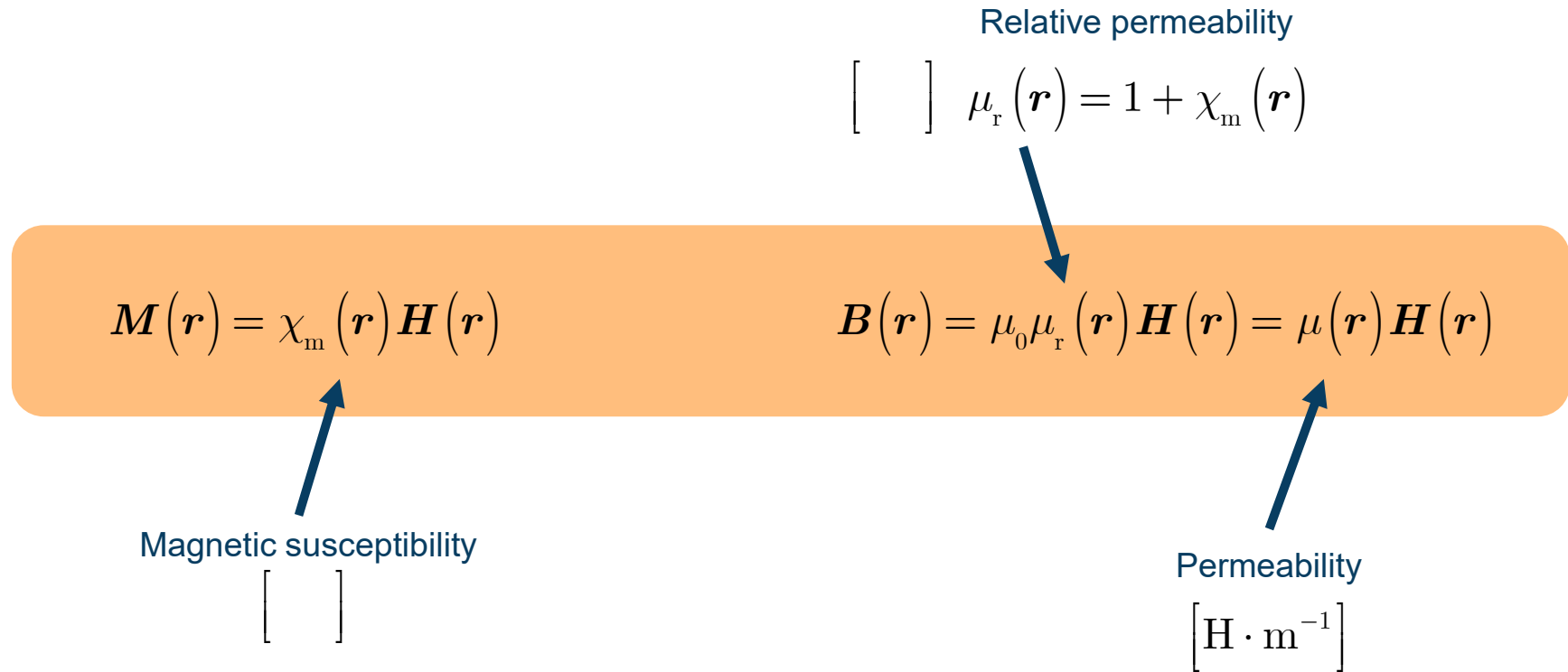
$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}(\mathbf{r})$$

$$\mathbf{B}(\mathbf{r}) = \mu_0 (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$

$$\oint_l \mathbf{H}(\mathbf{r}) \cdot d\mathbf{l} = I$$

Only free current

# Linear Isotropic Magnetic Materials



*All the complicated structure of matter reduces to a simple scalar quantity*

## Fields in Presence of Magnetic Material

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \Leftrightarrow \mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) \quad \Rightarrow \quad \nabla \times \left[ \frac{1}{\mu(\mathbf{r})} \nabla \times \mathbf{A}(\mathbf{r}) \right] = \mathbf{J}(\mathbf{r})$$

$$\Delta \mathbf{A}(\mathbf{r}) = -\mu \mathbf{J}(\mathbf{r})$$

$$\nabla \cdot \mathbf{A}(\mathbf{r}) = 0$$

Coulomb('s) gauge

Not a function of  
coordinates

*Poisson's equation holds only when permittivity does not depend on coordinates*

## Magnetic Material Boundaries

$$\mathbf{n}(\mathbf{r}) \times [\mathbf{H}_1(\mathbf{r}) - \mathbf{H}_2(\mathbf{r})] = \mathbf{K}(\mathbf{r})$$

$$\mathbf{n}(\mathbf{r}) \cdot [\mu_1 \mathbf{H}_1(\mathbf{r}) - \mu_2 \mathbf{H}_2(\mathbf{r})] = 0$$

Normal  
pointing to  
region (1)

*Both conditions are needed for unique solution*

## Magnetostatic Energy in Magnetic Material

$$W = \frac{1}{2\mu_0} \int_V |\mathbf{B}(\mathbf{r})|^2 dV \quad \Rightarrow \quad W = \frac{1}{2} \int_V \mathbf{H}(\mathbf{r}) \cdot \mathbf{B}(\mathbf{r}) dV$$

## Magnetic Materials

- **Paramagnetic** – small positive susceptibility  
(small attraction – **linear**)
- **Diamagnetic** – small negative susceptibility  
(small repulsion – **linear**)
- **Ferromagnetic** – “*large positive susceptibility*”  
(large attraction – **nonlinear**)



## Ferromagnetic Materials

- Spins are ordered within domains
- Magnetization is a non-linear function of field intensity
- Magnetization curve – Hysteresis, Remanence
- Susceptibility can only be defined as local approximation
- Above Curie('s) temperature ferromagnetism disappears

*Exact calculations are very difficult – use simplified models (soft material, permanent magnet)*

## Faraday('s) Law

Minus sign is called  
Lenz('s) law

$-\frac{\partial\Phi}{\partial t}$  Time variation of  
magnetic flux

$$\oint_l \mathbf{E}(\mathbf{r}, t) \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_s \mathbf{B}(\mathbf{r}, t) \cdot d\mathbf{S} \quad \longleftrightarrow \quad \nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t}$$

*Time variation in magnetic field produces electric field that tries to counter the change in magnetic flux (electromotive force)*

## Lenz('s) Law

The current created by time variation of magnetic flux is directed so as to oppose the flux creating it.

# Time Varying RL Circuits

In a loop

$$\sum_i U_i(t) = U_{\text{electromotive}}(t)$$

At a junction

$$\sum_i I_i(t) = 0$$

$$U_i(t) = R_i I_i(t)$$

On a resistor

$$U_1(t) = L_{11} \frac{\partial I_1(t)}{\partial t} + M_{12} \frac{\partial I_2(t)}{\partial t}$$

On an inductor

*Circuit laws are valid as long as the variations are not too fast*

## Time Varying Potentials

Potential  
calibration

$$\nabla \cdot \mathbf{A}(\mathbf{r}, t) = -\sigma\mu\varphi(\mathbf{r}, t)$$


$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\varphi(\mathbf{r}, t) - \frac{\partial\mathbf{A}(\mathbf{r}, t)}{\partial t}$$

*In time varying fields scalar potential becomes redundant*

## Source and Induced Currents

Those are fixed, not  
reacting to fields


$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \mathbf{J}_{\text{source}}(\mathbf{r}, t) + \mathbf{J}_{\text{induced}}(\mathbf{r}, t) = \mathbf{J}_{\text{source}}(\mathbf{r}, t) + \sigma \mathbf{E}(\mathbf{r}, t)$$

## Diffusion Equation

$$\Delta \mathbf{A}(\mathbf{r}, t) - \sigma \mu \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} = -\mu \mathbf{J}_{\text{source}}(\mathbf{r}, t)$$

$$\Delta \mathbf{H}(\mathbf{r}, t) - \sigma \mu \frac{\partial \mathbf{H}(\mathbf{r}, t)}{\partial t} = -\nabla \times \mathbf{J}_{\text{source}}(\mathbf{r}, t)$$

$$\Delta \mathbf{E}(\mathbf{r}, t) - \sigma \mu \frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t} = \frac{1}{\varepsilon} \nabla \rho_{\text{source}}(\mathbf{r}, t) + \mu \frac{\partial \mathbf{J}_{\text{source}}(\mathbf{r}, t)}{\partial t}$$

Material parameters are assumed independent of coordinates

## Maxwell('s) Equations

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \mathbf{J}(\mathbf{r}, t) + \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t}$$

Field equations

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = \rho(\mathbf{r}, t)$$

Force Law

$$\mathbf{f}(\mathbf{r}, t) = \rho(\mathbf{r}, t) \mathbf{E}(\mathbf{r}, t) + \mathbf{J}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t)$$

Interaction with materials

$$\begin{aligned} \mathbf{D}(\mathbf{r}, t) &= \epsilon_0 \mathbf{E}(\mathbf{r}, t) + \mathbf{P}(\mathbf{r}, t) \\ \mathbf{B}(\mathbf{r}, t) &= \mu_0 (\mathbf{H}(\mathbf{r}, t) + \mathbf{M}(\mathbf{r}, t)) \end{aligned}$$

*Absolute majority of things happening around you is described by these equations*



## Boundary Conditions


$$\mathbf{n}(\mathbf{r}) \times [\mathbf{E}_1(\mathbf{r}, t) - \mathbf{E}_2(\mathbf{r}, t)] = 0$$

$$\mathbf{n}(\mathbf{r}) \times [\mathbf{H}_1(\mathbf{r}, t) - \mathbf{H}_2(\mathbf{r}, t)] = \mathbf{K}(\mathbf{r}, t)$$

$$\mathbf{n}(\mathbf{r}) \cdot [\mathbf{B}_1(\mathbf{r}, t) - \mathbf{B}_2(\mathbf{r}, t)] = 0$$

$$\mathbf{n}(\mathbf{r}) \cdot [\mathbf{D}_1(\mathbf{r}, t) - \mathbf{D}_2(\mathbf{r}, t)] = \sigma(\mathbf{r}, t)$$

Normal  
pointing to  
region (1)



## Electromagnetic Potentials

Lorentz('s)  
calibration

$$\nabla \cdot \mathbf{A}(\mathbf{r}, t) = -\sigma\mu\varphi(\mathbf{r}, t) - \epsilon\mu \frac{\partial\varphi(\mathbf{r}, t)}{\partial t}$$

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\varphi(\mathbf{r}, t) - \frac{\partial\mathbf{A}(\mathbf{r}, t)}{\partial t}$$

## Wave Equation

$$\Delta \mathbf{A}(\mathbf{r}, t) - \sigma \mu \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} - \varepsilon \mu \frac{\partial^2 \mathbf{A}(\mathbf{r}, t)}{\partial t^2} = -\mu \mathbf{J}_{\text{source}}(\mathbf{r}, t)$$

Material parameters are assumed  
independent of coordinates

## Frequency Domain

$$\mathbf{F}(\mathbf{r}, t) \in \mathbb{R}$$

$$\hat{\mathbf{F}}(\mathbf{r}, \omega) \in \mathbb{C}$$

$$\mathbf{F}(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\mathbf{F}}(\mathbf{r}, \omega) e^{j\omega t} d\omega$$



$$\hat{\mathbf{F}}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} \mathbf{F}(\mathbf{r}, t) e^{-j\omega t} dt$$

$$\frac{\partial \mathbf{F}(\mathbf{r}, t)}{\partial t} \leftrightarrow j\omega \hat{\mathbf{F}}(\mathbf{r}, \omega)$$

Time derivatives reduce to algebraic multiplication

$$\frac{\partial \mathbf{F}(\mathbf{r}, t)}{\partial r_\xi} \leftrightarrow \frac{\partial \hat{\mathbf{F}}(\mathbf{r}, \omega)}{\partial r_\xi}$$

Spatial derivatives are untouched

*Frequency domain helps us to remove explicit time derivatives*

## Phasors

$$\hat{\mathbf{F}}(\mathbf{r}, -\omega) = \hat{\mathbf{F}}^*(\mathbf{r}, \omega)$$



$$\mathbf{F}(\mathbf{r}, t) = \frac{1}{\pi} \int_0^{\infty} \text{Re}[\hat{\mathbf{F}}(\mathbf{r}, \omega) e^{j\omega t}] d\omega$$

*Reduced frequency domain representation*

## Maxwell('s) Equations – Frequency Domain

$$\nabla \times \hat{\mathbf{H}}(\mathbf{r}, \omega) = \hat{\mathbf{J}}(\mathbf{r}, \omega) + j\omega\epsilon\hat{\mathbf{E}}(\mathbf{r}, \omega)$$

$$\nabla \times \hat{\mathbf{E}}(\mathbf{r}, \omega) = -j\omega\mu\hat{\mathbf{H}}(\mathbf{r}, \omega)$$

$$\nabla \cdot \hat{\mathbf{H}}(\mathbf{r}, \omega) = 0$$

$$\nabla \cdot \hat{\mathbf{E}}(\mathbf{r}, \omega) = \frac{\hat{\rho}(\mathbf{r}, \omega)}{\epsilon}$$

*We assume linearity of material relations*

## Wave Equation – Frequency Domain

$$\Delta \hat{\mathbf{A}}(\mathbf{r}, \omega) - j\omega\mu(\sigma + j\omega\varepsilon)\hat{\mathbf{A}}(\mathbf{r}, \omega) = -\mu\hat{\mathbf{J}}_{\text{source}}(\mathbf{r}, \omega)$$

Helmholtz('s) equation

## Plane Wave

Unitary vector representing the direction of propagation

$$\hat{\mathbf{E}}(\mathbf{r}, \omega) = \mathbf{E}_0(\omega) e^{-j\mathbf{k}\mathbf{n}\cdot\mathbf{r}}$$

Electric and magnetic fields are mutually orthogonal

$$\hat{\mathbf{H}}(\mathbf{r}, \omega) = \frac{k}{\omega\mu} [\mathbf{n} \times \mathbf{E}_0(\omega)] e^{-j\mathbf{k}\mathbf{n}\cdot\mathbf{r}}$$

Electric and magnetic fields are orthogonal to propagation direction

$$\mathbf{n} \cdot \mathbf{E}_0(\omega) = 0$$
$$\mathbf{n} \cdot \mathbf{H}_0(\omega) = 0$$

Wave-number

$$k^2 = -j\omega\mu(\sigma + j\omega\varepsilon)$$

*The simplest wave solution of Maxwell('s) equations*



## Plane Wave Characteristics

$$k = \sqrt{-j\omega\mu(\sigma + j\omega\varepsilon)}$$

$$\operatorname{Re}[k] > 0; \operatorname{Im}[k] < 0$$

$$\lambda = \frac{2\pi}{\operatorname{Re}[k]}$$

$$v_f = \frac{\omega}{\operatorname{Re}[k]}$$

$$Z = \frac{\omega\mu}{k}$$

$$\delta = -\frac{1}{\operatorname{Im}[k]}$$

Vacuum



$$k = \frac{\omega}{c_0}$$

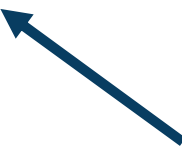
$$\operatorname{Re}[k] > 0; \operatorname{Im}[k] = 0$$

$$\lambda = \frac{c_0}{f}$$

$$v_f = c_0$$

$$Z = c_0\mu_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377 \Omega$$

$$\delta \rightarrow \infty$$



General isotropic material

## Poynting('s)-Umov('s) Theorem

Power passing the bounding envelope

Energy storage

$$-\int_V \mathbf{E} \cdot \mathbf{J}_{\text{source}} dV = \oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} + \int_V \sigma |\mathbf{E}|^2 dV + \frac{1}{2} \frac{\partial}{\partial t} \int_V (\epsilon |\mathbf{E}|^2 + \mu |\mathbf{H}|^2) dV$$

Power supplied by sources

Heat losses

*Energy balance in an electromagnetic system*

## Heat Balance in Time-Harmonic Steady State

Valid for general periodic steady state


Cycle mean

$$-\int_V \langle \mathbf{E} \cdot \mathbf{J}_{\text{source}} \rangle dV = \oint_S \langle \mathbf{E} \times \mathbf{H} \rangle \cdot d\mathbf{S} + \int_V \langle \sigma |\mathbf{E}|^2 \rangle dV$$
$$-\frac{1}{2} \int_V \text{Re}[\hat{\mathbf{E}} \cdot \hat{\mathbf{J}}_{\text{source}}^*] dV = \frac{1}{2} \oint_S \text{Re}[\hat{\mathbf{E}} \times \hat{\mathbf{H}}^*] \cdot d\mathbf{S} + \frac{1}{2} \int_V \sigma |\hat{\mathbf{E}}|^2 dV$$

Valid for time-harmonic steady state


## Cycle Mean Power Density of a Plane Wave

Power propagation coincides with phase propagation


$$\langle \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \rangle = \frac{1}{2} \frac{\text{Re}[k]}{\omega \mu} |\mathbf{E}_0(\omega)|^2 e^{2\text{Im}[k] \cdot \mathbf{r}} \mathbf{n}$$

## Linear Momentum Carried by Fields

Volume integration considerably change  
the meaning of Poynting('s) vector


$$\mathbf{p} = \frac{1}{c_0^2} \int_V (\mathbf{E} \times \mathbf{H}) dV$$

*This formula is only valid in vacuum. In material media things are more tricky.*

## Angular Momentum Carried by Fields

$$\mathbf{L} = \frac{1}{c_0^2} \int_V \mathbf{r} \times (\mathbf{E} \times \mathbf{H}) dV$$

*This formula is only valid in vacuum. In material media things are more tricky.*



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