



Electromagnetic Field Theory 2

(fundamental relations and definitions)

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Maxwell('s)-Lorentz('s) Equations

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \mathbf{J}(\mathbf{r}, t) + \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t}$$

Equations of motion
for fields

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = \rho(\mathbf{r}, t)$$

Equation of motion
for particles

$$\mathbf{f}(\mathbf{r}, t) = \rho(\mathbf{r}, t) \mathbf{E}(\mathbf{r}, t) + \mathbf{J}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t)$$

Interaction with materials

$$\mathbf{D}(\mathbf{r}, t) = \varepsilon_0 \mathbf{E}(\mathbf{r}, t) + \mathbf{P}(\mathbf{r}, t)$$

$$\mathbf{B}(\mathbf{r}, t) = \mu_0 (\mathbf{H}(\mathbf{r}, t) + \mathbf{M}(\mathbf{r}, t))$$

Absolute majority of things happening around us is described by these equations



Boundary Conditions

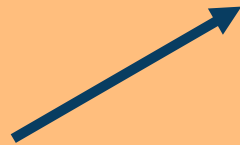
$$\mathbf{n}(\mathbf{r}) \times [\mathbf{E}_1(\mathbf{r}, t) - \mathbf{E}_2(\mathbf{r}, t)] = 0$$

$$\mathbf{n}(\mathbf{r}) \times [\mathbf{H}_1(\mathbf{r}, t) - \mathbf{H}_2(\mathbf{r}, t)] = \mathbf{K}(\mathbf{r}, t)$$

$$\mathbf{n}(\mathbf{r}) \cdot [\mathbf{B}_1(\mathbf{r}, t) - \mathbf{B}_2(\mathbf{r}, t)] = 0$$

$$\mathbf{n}(\mathbf{r}) \cdot [\mathbf{D}_1(\mathbf{r}, t) - \mathbf{D}_2(\mathbf{r}, t)] = \sigma(\mathbf{r}, t)$$

Normal
pointing to
region (1)



Electromagnetic Potentials

Lorentz('s)
calibration

$$\nabla \cdot \mathbf{A}(\mathbf{r}, t) = -\sigma\mu\varphi(\mathbf{r}, t) - \varepsilon\mu \frac{\partial\varphi(\mathbf{r}, t)}{\partial t}$$

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\varphi(\mathbf{r}, t) - \frac{\partial\mathbf{A}(\mathbf{r}, t)}{\partial t}$$



Wave Equation

$$\Delta \mathbf{A}(\mathbf{r}, t) - \sigma \mu \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} - \varepsilon \mu \frac{\partial^2 \mathbf{A}(\mathbf{r}, t)}{\partial t^2} = -\mu \mathbf{J}_{\text{source}}(\mathbf{r}, t)$$

Material parameters are assumed independent of coordinates



Poynting('s)-Umov('s) Theorem

Power passing the bounding envelope

Energy storage

$$-\int_V \mathbf{E} \cdot \mathbf{J}_{\text{source}} dV = \oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} + \int_V \sigma |\mathbf{E}|^2 dV + \frac{1}{2} \frac{\partial}{\partial t} \int_V (\epsilon |\mathbf{E}|^2 + \mu |\mathbf{H}|^2) dV$$

Power supplied by sources

Heat losses

Energy balance in an electromagnetic system



Frequency Domain

$$\mathbf{F}(\mathbf{r}, t) \in \mathbb{R}$$

$$\hat{\mathbf{F}}(\mathbf{r}, \omega) \in \mathbb{C}$$

$$\mathbf{F}(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\mathbf{F}}(\mathbf{r}, \omega) e^{j\omega t} d\omega$$



$$\hat{\mathbf{F}}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} \mathbf{F}(\mathbf{r}, t) e^{-j\omega t} dt$$

$$\frac{\partial \mathbf{F}(\mathbf{r}, t)}{\partial t} \leftrightarrow j\omega \hat{\mathbf{F}}(\mathbf{r}, \omega)$$

Time derivatives reduce to algebraic multiplication

$$\frac{\partial \mathbf{F}(\mathbf{r}, t)}{\partial r_\xi} \leftrightarrow \frac{\partial \hat{\mathbf{F}}(\mathbf{r}, \omega)}{\partial r_\xi}$$

Spatial derivatives are untouched

Frequency domain helps us to remove explicit time derivatives



Phasors

$$\hat{\mathbf{F}}(\mathbf{r}, -\omega) = \hat{\mathbf{F}}^*(\mathbf{r}, \omega)$$



$$\mathbf{F}(\mathbf{r}, t) = \frac{1}{\pi} \int_0^{\infty} \text{Re}[\hat{\mathbf{F}}(\mathbf{r}, \omega) e^{j\omega t}] d\omega$$

Reduced frequency domain representation



Maxwell('s) Equations – Frequency Domain

$$\nabla \times \hat{\mathbf{H}}(\mathbf{r}, \omega) = \hat{\mathbf{J}}(\mathbf{r}, \omega) + j\omega\epsilon\hat{\mathbf{E}}(\mathbf{r}, \omega)$$

$$\nabla \times \hat{\mathbf{E}}(\mathbf{r}, \omega) = -j\omega\mu\hat{\mathbf{H}}(\mathbf{r}, \omega)$$

$$\nabla \cdot \hat{\mathbf{H}}(\mathbf{r}, \omega) = 0$$

$$\nabla \cdot \hat{\mathbf{E}}(\mathbf{r}, \omega) = \frac{\hat{\rho}(\mathbf{r}, \omega)}{\epsilon}$$

We assume linearity of material relations



Wave Equation – Frequency Domain

$$\Delta \hat{\mathbf{A}}(\mathbf{r}, \omega) - j\omega\mu(\sigma + j\omega\varepsilon)\hat{\mathbf{A}}(\mathbf{r}, \omega) = -\mu\hat{\mathbf{J}}_{\text{source}}(\mathbf{r}, \omega)$$

Helmholtz('s) equation



Heat Balance in Time-Harmonic Steady State

Valid for general periodic steady state

Cycle mean

$$-\int_V \langle \mathbf{E} \cdot \mathbf{J}_{\text{source}} \rangle dV = \oint_S \langle \mathbf{E} \times \mathbf{H} \rangle \cdot d\mathbf{S} + \int_V \langle \sigma |\mathbf{E}|^2 \rangle dV$$
$$-\frac{1}{2} \int_V \text{Re}[\hat{\mathbf{E}} \cdot \hat{\mathbf{J}}_{\text{source}}^*] dV = \frac{1}{2} \oint_S \text{Re}[\hat{\mathbf{E}} \times \hat{\mathbf{H}}^*] \cdot d\mathbf{S} + \frac{1}{2} \int_V \sigma |\hat{\mathbf{E}}|^2 dV$$

Valid for time-harmonic steady state



Plane Wave

Unitary vector representing the direction of propagation

Electric and magnetic fields are mutually orthogonal

Electric and magnetic fields are orthogonal to propagation direction

Wave-number

$$\hat{\mathbf{E}}(\mathbf{r}, \omega) = \mathbf{E}_0(\omega) e^{-j\mathbf{k}\mathbf{n}\cdot\mathbf{r}}$$
$$\hat{\mathbf{H}}(\mathbf{r}, \omega) = \frac{k}{\omega\mu} [\mathbf{n} \times \mathbf{E}_0(\omega)] e^{-j\mathbf{k}\mathbf{n}\cdot\mathbf{r}}$$
$$\mathbf{n} \cdot \mathbf{E}_0(\omega) = 0$$
$$\mathbf{n} \cdot \mathbf{H}_0(\omega) = 0$$
$$k^2 = -j\omega\mu(\sigma + j\omega\varepsilon)$$

The simplest wave solution of Maxwell('s) equations



Plane Wave Characteristics

$$k = \sqrt{-j\omega\mu(\sigma + j\omega\varepsilon)}$$

$$\operatorname{Re}[k] > 0; \operatorname{Im}[k] < 0$$

$$\lambda = \frac{2\pi}{\operatorname{Re}[k]}$$

$$v_f = \frac{\omega}{\operatorname{Re}[k]}$$

$$Z = \frac{\omega\mu}{k}$$

$$\delta = -\frac{1}{\operatorname{Im}[k]}$$

Vacuum



$$k = \frac{\omega}{c_0}$$

$$\operatorname{Re}[k] > 0; \operatorname{Im}[k] = 0$$

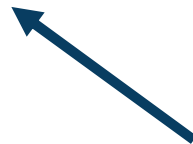
$$\lambda = \frac{c_0}{f}$$

$$v_f = c_0$$

$$Z = c_0\mu_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377 \Omega$$

$$\delta \rightarrow \infty$$

General isotropic material



Cycle Mean Power Density of a Plane Wave

Power propagation coincides with phase propagation

$$\langle \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \rangle = \frac{1}{2} \frac{\operatorname{Re}[k]}{\omega \mu} |\mathbf{E}_0(\omega)|^2 e^{2\operatorname{Im}[k] \cdot \mathbf{n} \cdot \mathbf{r}} \mathbf{n}$$



Source Free Maxwell('s) Equations in Free Space

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \sigma(t) * \mathbf{E}(\mathbf{r}, t) + \varepsilon(t) * \frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t}$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\mu(t) * \frac{\partial \mathbf{H}(\mathbf{r}, t)}{\partial t}$$

$$\nabla \cdot \mathbf{H}(\mathbf{r}, t) = 0$$

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = 0$$



$$|\mathbf{k}|^2 = k^2 = -j\omega\hat{\mu}(\omega)(\hat{\sigma}(\omega) + j\omega\hat{\varepsilon}(\omega))$$

$$\hat{\mathbf{E}}(\mathbf{k}, \omega) = \frac{\mathbf{k}}{\omega\hat{\varepsilon}(\omega) - j\hat{\sigma}(\omega)} \times \hat{\mathbf{H}}(\mathbf{k}, \omega)$$

$$\hat{\mathbf{H}}(\mathbf{k}, \omega) = -\frac{\mathbf{k}}{\omega\hat{\mu}(\omega)} \times \hat{\mathbf{E}}(\mathbf{k}, \omega)$$

$$\mathbf{k} \cdot \hat{\mathbf{E}}(\mathbf{k}, \omega) = 0$$

$$\mathbf{k} \cdot \hat{\mathbf{H}}(\mathbf{k}, \omega) = 0$$

$$\mathbf{F}(\mathbf{r}, t) = \frac{1}{(2\pi)^4} \int_{\mathbf{k}, t} \hat{\mathbf{F}}(\mathbf{k}, \omega) e^{j(\mathbf{k} \cdot \mathbf{r} + \omega t)} d\mathbf{k} d\omega$$

Fourier's transform leads to simple algebraic equations



Spatial Wave Packet

$$|\mathbf{k}|^2 = k^2 = -j\omega\hat{\mu}(\omega)(\hat{\sigma}(\omega) + j\omega\hat{\epsilon}(\omega)) \quad \longrightarrow \quad \omega = \omega(|\mathbf{k}|)$$

This can be electric or magnetic intensity

$$\mathbf{F}(\mathbf{r}, t) = \frac{1}{(2\pi)^3} \int_{\mathbf{k}} \hat{\mathbf{F}}_0(\mathbf{k}) e^{j(\mathbf{k}\cdot\mathbf{r} + \omega(|\mathbf{k}|)t)} d\mathbf{k}$$
$$\mathbf{k} \cdot \hat{\mathbf{F}}_0(\mathbf{k}) = 0$$

General solution to free-space Maxwell's equations



Spatial Wave Packet in Vacuum

$$\omega(|\mathbf{k}|) = \pm c_0 |\mathbf{k}|$$

$$\mathbf{k} \cdot \hat{\mathbf{F}}_0^+(\mathbf{k}) = \mathbf{k} \cdot \hat{\mathbf{F}}_0^-(\mathbf{k}) = 0$$

$$\hat{\mathbf{F}}_0^-(\mathbf{k}) = [\hat{\mathbf{F}}_0^+(-\mathbf{k})]^*$$

$$\hat{\mathbf{F}}_0^+(\mathbf{k}) = [\hat{\mathbf{F}}_0^-(-\mathbf{k})]^*$$

$$\mathbf{F}(\mathbf{r}, t) = \frac{1}{(2\pi)^3} \int_{\mathbf{k}} e^{j\mathbf{k}\cdot\mathbf{r}} \left[\hat{\mathbf{F}}_0^+(\mathbf{k}) e^{j c_0 t |\mathbf{k}|} + \hat{\mathbf{F}}_0^-(\mathbf{k}) e^{-j c_0 t |\mathbf{k}|} \right] d\mathbf{k}$$

$$\hat{\mathbf{F}}_0^+(\mathbf{k}) = \frac{1}{2} \int_{\mathbf{r}} \left[\mathbf{F}(\mathbf{r}, 0) + \frac{1}{j c_0 |\mathbf{k}|} \left. \frac{\partial \mathbf{F}(\mathbf{r}, t)}{\partial t} \right|_{t=0} \right] e^{-j\mathbf{k}\cdot\mathbf{r}} d\mathbf{r}$$

$$\hat{\mathbf{F}}_0^-(\mathbf{k}) = \frac{1}{2} \int_{\mathbf{r}} \left[\mathbf{F}(\mathbf{r}, 0) - \frac{1}{j c_0 |\mathbf{k}|} \left. \frac{\partial \mathbf{F}(\mathbf{r}, t)}{\partial t} \right|_{t=0} \right] e^{-j\mathbf{k}\cdot\mathbf{r}} d\mathbf{r}$$

The field is uniquely given by initial conditions



Spatial Wave Packet in Vacuum

$$\omega(|\mathbf{k}|) = \pm c_0 |\mathbf{k}|$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{(2\pi)^3} \int_{\mathbf{k}} e^{j\mathbf{k}\cdot\mathbf{r}} \left[\hat{\mathbf{E}}^+(\mathbf{k}) e^{j c_0 t |\mathbf{k}|} + \hat{\mathbf{E}}^-(\mathbf{k}) e^{-j c_0 t |\mathbf{k}|} \right] d\mathbf{k}$$

$$\mathbf{H}(\mathbf{r}, t) = -\frac{1}{(2\pi)^3} \int_{\mathbf{k}} e^{j\mathbf{k}\cdot\mathbf{r}} \frac{\mathbf{k}}{Z_0 |\mathbf{k}|} \times \left[\hat{\mathbf{E}}^+(\mathbf{k}) e^{j c_0 t |\mathbf{k}|} - \hat{\mathbf{E}}^-(\mathbf{k}) e^{-j c_0 t |\mathbf{k}|} \right] d\mathbf{k}$$

$$\mathbf{k} \cdot \hat{\mathbf{E}}^+(\mathbf{k}) = \mathbf{k} \cdot \hat{\mathbf{E}}^-(\mathbf{k}) = 0$$

Electric and magnetic field are not independent



Vacuum Dispersion

1D problem

$$\hat{\mathbf{E}}^+(\mathbf{k}) = \hat{\mathbf{E}}^-(\mathbf{k}) = (2\pi)^2 \hat{\mathbf{E}}^+(k_z) \delta(k_x) \delta(k_y)$$

$$\mathbf{z}_0 \cdot \hat{\mathbf{E}}^+(k_z) = \mathbf{z}_0 \cdot \hat{\mathbf{E}}^-(k_z) = 0$$

Propagation in one direction

no dispersion

$$\mathbf{E}(z, t) = \mathbf{E}^+(z + c_0 t) + \mathbf{E}^-(z - c_0 t)$$

$$\mathbf{H}(z, t) = -\frac{1}{Z_0} \mathbf{z}_0 \times [\mathbf{E}^+(z + c_0 t) - \mathbf{E}^-(z - c_0 t)]$$

1D waves in vacuum propagate without dispersion



Vacuum Dispersion

In general this term does not represent translation

$$([x, y, z] \pm c_0 t)$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{(2\pi)^3} \int_{\mathbf{k}} e^{j\mathbf{k} \cdot \mathbf{r}} \left[\hat{\mathbf{E}}^+(\mathbf{k}) e^{j c_0 t |\mathbf{k}|} + \hat{\mathbf{E}}^-(\mathbf{k}) e^{-j c_0 t |\mathbf{k}|} \right] d\mathbf{k}$$

Waves propagating in all directions

2D and 3D waves in vacuum always disperse = change shape in time



Angular Spectrum Representation

$$\text{Im}[k_z] < 0$$

$$|\mathbf{k}|^2 = k^2 = -j\omega\hat{\mu}(\omega)(\hat{\sigma}(\omega) + j\omega\hat{\epsilon}(\omega)) \quad \Rightarrow \quad k_z = \pm\sqrt{k^2 - k_x^2 - k_y^2}$$

$$\hat{\mathbf{H}}_0(k_x, k_y, \omega) = -\frac{\mathbf{k}}{Z|\mathbf{k}|} \times \hat{\mathbf{E}}_0(k_x, k_y, \omega)$$

$$\hat{\mathbf{E}}_0(k_x, k_y, \omega) = \mathcal{F}_{x,y,t} \{ \mathbf{E}(x, y, 0, t) \}$$

$$\mathbf{E}(x, y, z < 0, t) = \frac{1}{(2\pi)^3} \int_{k_x, k_y, \omega} e^{j(k_x x + k_y y + \omega t)} \hat{\mathbf{E}}_0(k_x, k_y, \omega) e^{j\sqrt{k^2 - k_x^2 - k_y^2} z} dk_x dk_y d\omega$$

$$\mathbf{E}(x, y, z > 0, t) = \frac{1}{(2\pi)^3} \int_{k_x, k_y, \omega} e^{j(k_x x + k_y y + \omega t)} \hat{\mathbf{E}}_0(k_x, k_y, \omega) e^{-j\sqrt{k^2 - k_x^2 - k_y^2} z} dk_x dk_y d\omega$$

$$\mathbf{k} \cdot \hat{\mathbf{E}}_0 = 0$$

General solution to free-space Maxwell's equations



Propagating vs Evanescent Waves

$$k_x^2 + k_y^2 < k^2$$

These waves propagate and
can carry information to far distances

$$k_x^2 + k_y^2 > k^2$$

These waves exponentially decay in amplitude
and cannot carry information to far distances

Field picture losses its resolution with distance from the source plane



Paraxial Waves

$$\hat{\mathbf{E}}_0(k_x, k_y, \omega) \quad \Rightarrow \quad k_x^2 + k_y^2 \ll k^2 \quad \Rightarrow \quad \sqrt{k^2 - k_x^2 - k_y^2} \approx k - \frac{1}{2k}(k_x^2 + k_y^2)$$

$$\mathbf{E}(x, y, z > 0, t) = \frac{1}{(2\pi)^3} \int_{k_x, k_y, \omega} e^{j(k_x x + k_y y - kz + \omega t)} \hat{\mathbf{E}}_0(k_x, k_y, \omega) e^{j\frac{1}{2k}(k_x^2 + k_y^2)z} dk_x dk_y d\omega$$

$$\mathbf{k} \cdot \hat{\mathbf{E}}_0 = 0$$

Propagates almost as a planewave

$$\mathbf{z}_0 \cdot \hat{\mathbf{E}}_0 \approx 0$$



Gaussian Beam

$$\hat{\mathbf{E}}_{0\perp}(k_x, k_y, \omega) = \mathbf{A}_{0\perp} \pi w_0^2 e^{-\frac{1}{4} w_0^2 (k_x^2 + k_y^2)}$$



$$\mathbf{E}_{\perp}(x, y, z > 0, t) = \frac{1}{2\pi} \int_{\omega} \mathbf{A}_{0\perp} \frac{w_0}{w(z)} e^{-\frac{x^2 + y^2}{w^2(z)}} e^{j \arctan\left(\frac{z}{z_R}\right)} e^{-j \frac{x^2 + y^2}{w^2(z)} \left[\frac{z}{z_R}\right]} e^{j\omega \left[t - \frac{z}{c_0}\right]} d\omega$$

Gaussian profile
in amplitude

Paraxial
corrections

Planewave-like
propagation

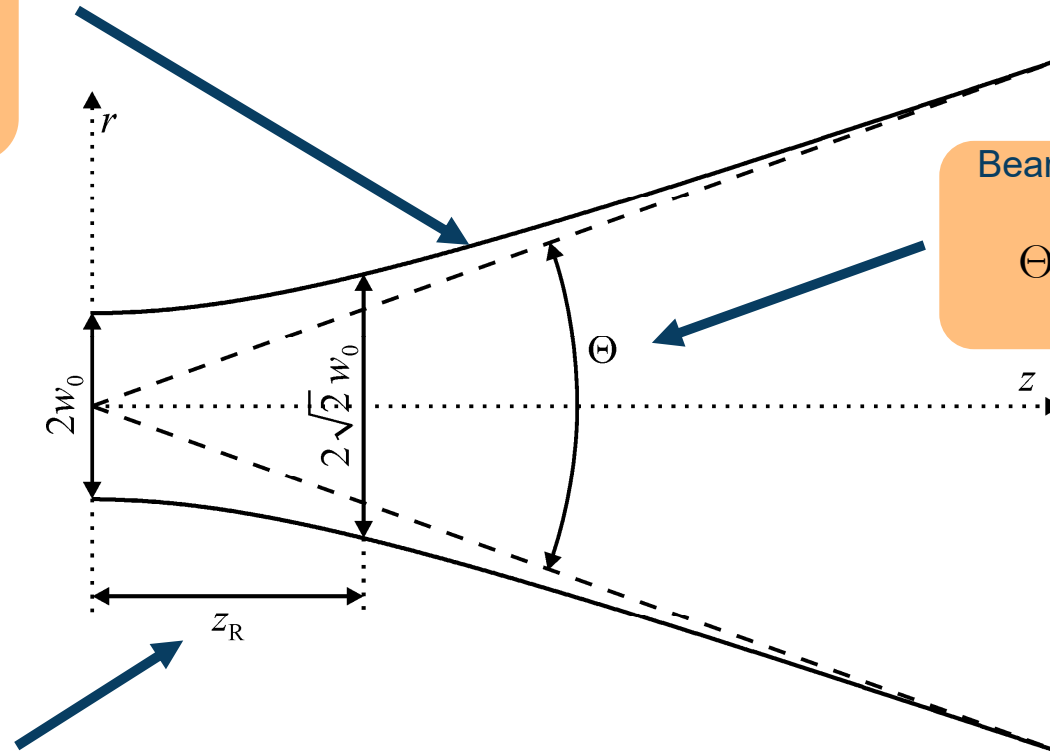
Approximates radiation of sources large in comparison to wavelength



Gaussian Beam

Half-width of the beam

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2}$$



Beam divergence

$$\Theta = \frac{2\lambda}{\pi w_0}$$

Rayleigh's distance

$$z_R = \frac{1}{2} k w_0^2 = \frac{\pi w_0^2}{\lambda}$$



Gaussian Beam – Time-Harmonic Case

$$\langle \mathbf{S} \rangle = \frac{1}{2} \operatorname{Re} \left[\hat{\mathbf{E}}(x, y, z, \omega) \times \hat{\mathbf{H}}^*(x, y, z, \omega) \right] = \mathbf{z}_0 S_0 \frac{w_0^2}{w^2(z)} e^{-\frac{2\rho^2}{w^2(z)}}$$

86.5 % of power flows through the beam width

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_R} \right)^2}$$

Power density at origin



Material Dispersion

Causality requirement

$$\varepsilon(\tau) = 0, \tau < 0$$

Stability requirement

$$\varepsilon(\tau) \rightarrow 0, \tau \rightarrow \infty$$

$$\mathbf{D}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \varepsilon(\tau) \mathbf{E}(\mathbf{r}, t - \tau) d\tau$$

$$\mathbf{B}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \mu(\tau) \mathbf{H}(\mathbf{r}, t - \tau) d\tau$$

$$\mathbf{J}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \sigma(\tau) \mathbf{E}(\mathbf{r}, t - \tau) d\tau$$



$$\hat{\mathbf{D}}(\mathbf{r}, \omega) = \hat{\varepsilon}(\omega) \hat{\mathbf{E}}(\mathbf{r}, \omega)$$

$$\hat{\mathbf{B}}(\mathbf{r}, \omega) = \hat{\mu}(\omega) \hat{\mathbf{H}}(\mathbf{r}, \omega)$$

$$\hat{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\sigma}(\omega) \hat{\mathbf{E}}(\mathbf{r}, \omega)$$

Even single planewave undergoes time dispersion when materials are present



Lorentz's Dispersion Model

Losses Resonance Coupling strength

$$\frac{\partial^2 \mathbf{P}(t)}{\partial t^2} + \Gamma \frac{\partial \mathbf{P}(t)}{\partial t} + \omega_0^2 \mathbf{P}(t) = \varepsilon_0 \omega_p^2 \mathbf{E}(t)$$

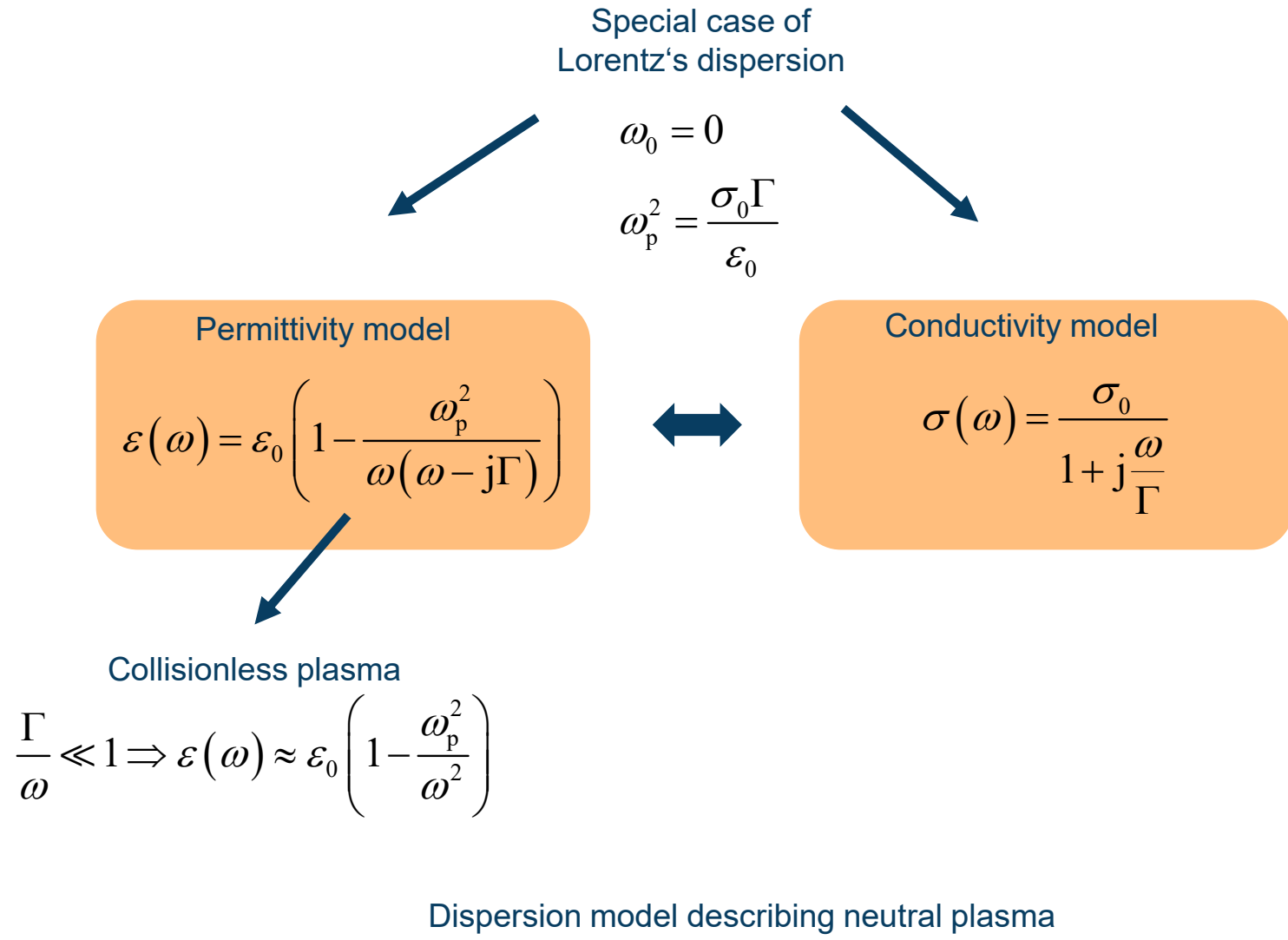
$$\varepsilon(\omega) = \varepsilon_0 \left(1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 + j\omega\Gamma} \right)$$

$$\varepsilon(\omega) = \varepsilon_0 \left(1 + \sum_i \frac{\omega_{p,i}^2}{\omega_{0,i}^2 - \omega^2 + j\omega\Gamma_i} \right)$$

Dispersion model able to describe vast amount of natural materials



Drude's Dispersion Model



Appleton's Dispersion Model

$$\frac{\partial^2 \mathbf{P}(t)}{\partial t^2} + \omega_c \frac{\partial \mathbf{P}(t)}{\partial t} \times \mathbf{z}_0 = \varepsilon_0 \omega_p^2 \mathbf{E}(t)$$

Cyclotron frequency

Direction of magnetization

Plasma frequency

$\hat{\varepsilon} \neq \hat{\varepsilon}^T$
 Propagation in opposite directions is not the same

$$\hat{\varepsilon} = \varepsilon_0 \begin{bmatrix} 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} & \frac{-j\omega_c \omega_p^2}{\omega(\omega^2 - \omega_c^2)} & 0 \\ \frac{j\omega_c \omega_p^2}{\omega(\omega^2 - \omega_c^2)} & 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_p^2}{\omega^2} \end{bmatrix}$$

Dispersion model describing magnetized neutral plasma



Propagation in Appleton's Dispersion Model

$$\hat{\mathbf{E}} = \mathbf{E}_0 e^{jk_z z}$$
$$\mathbf{k} \cdot \mathbf{E}_0 = 0$$

Planewave propagation
along magnetization

Fundamental modes
are circularly polarized
waves

$$\frac{k_z^2}{k_0^2} = 1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_c)}$$

$$\hat{E}_x = \mp j \hat{E}_y$$

Dispersion model describing magnetized neutral plasma



Radiation

Microscopic
charge velocity

$$\frac{\partial \mathbf{v}(t)}{\partial t} \neq 0$$

Macroscopic
current density

$$\frac{\partial \mathbf{J}(\mathbf{r}, t)}{\partial t} \neq 0$$



Any surface
circumscribing
the sources

$$\int_{-\infty}^{\infty} \oint_S (\mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t)) \cdot d\mathbf{S} dt \neq 0$$

Only accelerating charges can radiate



Time-Harmonic Electric Dipole

$$\hat{\mathbf{P}}(\mathbf{r}, \omega) = \mathbf{z}_0 p_z(\omega) \delta(x) \delta(y) \delta(z)$$

$\rho(\mathbf{r}, \omega) \approx 0$

$$\hat{\mathbf{A}}(\mathbf{r}, \omega) = jZ_0 k^2 (\mathbf{r}_0 \cos \theta - \theta_0 \sin \theta) p_z(\omega) \frac{e^{-jkr}}{4\pi kr}$$

$$\hat{\mathbf{H}}(\mathbf{r}, \omega) = c_0 k^3 \varphi_0 \sin \theta \left(-1 + \frac{j}{kr} \right) p_z(\omega) \frac{e^{-jkr}}{4\pi kr}$$

$$\hat{\mathbf{E}}(\mathbf{r}, \omega) = Z_0 c_0 k^3 \left[2\mathbf{r}_0 \cos \theta \left(\frac{j}{kr} + \frac{1}{k^2 r^2} \right) + \theta_0 \left(-1 + \frac{j}{kr} + \frac{1}{k^2 r^2} \right) \sin \theta \right] p_z(\omega) \frac{e^{-jkr}}{4\pi kr}$$

Elementary source of radiation



Time-Harmonic Electric Dipole - Field Zones

Static zone

$$\hat{\mathbf{E}}(\mathbf{r}, \omega) = Z_0 c_0 k^3 \left[2\mathbf{r}_0 \cos \theta \left(\frac{j}{kr} + \frac{1}{k^2 r^2} \right) + \theta_0 \left(-1 + \frac{j}{kr} + \frac{1}{k^2 r^2} \right) \sin \theta \right] p_z(\omega) \frac{e^{-jkr}}{4\pi kr}$$

Induction zone

Radiation zone

Static, quasi-static and fully dynamic terms all appear in the formula



Time-Harmonic Electric Dipole - Radiation Zone

$$\hat{\mathbf{P}}(\mathbf{r}, \omega) = \mathbf{z}_0 p_z(\omega) \delta(x) \delta(y) \delta(z)$$



$$\hat{\mathbf{E}}_\infty(\mathbf{r}, \omega) \approx -Z_0 c_0 k^3 \theta_0 p_z(\omega) \frac{e^{-jkr}}{4\pi kr} \sin \theta$$

$$\hat{\mathbf{H}}_\infty(\mathbf{r}, \omega) \approx \frac{1}{Z_0} \mathbf{r}_0 \times \hat{\mathbf{E}}_\infty(\mathbf{r}, \omega)$$

$$\langle \mathbf{S}_\infty \rangle = \frac{1}{2} \text{Re} [\hat{\mathbf{E}}_\infty \times \hat{\mathbf{H}}_\infty^*] = \frac{1}{2Z_0} |\hat{\mathbf{E}}_\infty(\mathbf{r}, \omega)|^2 \mathbf{r}_0$$

$$\mathbf{r}_0 \cdot \hat{\mathbf{E}}_\infty \approx 0$$

Radiated power [W]

$$P_{\text{rad}} = \frac{c_0^2 Z_0 k^4}{12\pi} |p_z(\omega)|^2$$

Farfield has a planewave-like geometry



Time-Harmonic Electric Dipole – General Case

$$\hat{\mathbf{P}}(\mathbf{r}, \omega) = \hat{\mathbf{p}}(\omega) \delta(\mathbf{r} - \mathbf{r}')$$



$$\mathbf{R} = \mathbf{r} - \mathbf{r}'$$

$$\hat{\mathbf{H}}(\mathbf{r}, \omega) = c_0 k^3 \left(\frac{\mathbf{R}}{R} \times \hat{\mathbf{p}} \right) \left(1 + \frac{1}{jkR} \right) \frac{e^{-jkR}}{4\pi kR}$$

$$\hat{\mathbf{E}}(\mathbf{r}, \omega) = Z_0 c_0 k^3 \left[-\frac{\mathbf{R}}{R} \times \left(\frac{\mathbf{R}}{R} \times \hat{\mathbf{p}} \right) + \left(3 \frac{\mathbf{R}}{R} \left[\hat{\mathbf{p}} \cdot \frac{\mathbf{R}}{R} \right] - \hat{\mathbf{p}} \right) \left(\frac{1}{k^2 R^2} + \frac{j}{kR} \right) \right] \frac{e^{-jkR}}{4\pi kR}$$

$$R = |\mathbf{r} - \mathbf{r}'|$$

Elementary source of radiation



General Radiator

$$\hat{\mathbf{J}}(\mathbf{r}, \omega), \mathbf{J}(\mathbf{r}, t)$$



time
retardation

$$\hat{\mathbf{A}}(\mathbf{r}, \omega) = \frac{\mu_0}{4\pi} \int_{V'} \hat{\mathbf{J}}(\mathbf{r}', \omega) \frac{e^{-jk_0 R}}{R} dV'$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}\left(\mathbf{r}', t - \frac{R}{c_0}\right)}{R} dV'$$

$$R = |\mathbf{r} - \mathbf{r}'|$$

Superposition of dipole fields



Field in Radiation Zone – General Case FD

$$kR \gg 1 \quad \wedge \quad r \gg r'$$

$$\hat{\mathbf{A}}_{\infty}(\mathbf{r}, \omega) \approx \frac{\mu_0}{4\pi} \frac{e^{-jk_0 r}}{r} \int_{V'} \hat{\mathbf{J}}(\mathbf{r}', \omega) e^{jk_0 \mathbf{r}_0 \cdot \mathbf{r}'} dV'$$

$$\hat{\mathbf{H}}_{\infty}(\mathbf{r}, \omega) \approx -\frac{j\omega}{Z_0} \mathbf{r}_0 \times \hat{\mathbf{A}}_{\infty}(\mathbf{r}, \omega)$$

$$\hat{\mathbf{E}}_{\infty} \approx -Z_0 (\mathbf{r}_0 \times \hat{\mathbf{H}}_{\infty})$$

$$\hat{\mathbf{E}}_{\infty}(\mathbf{r}, \omega) \approx j\omega \mathbf{r}_0 \times (\mathbf{r}_0 \times \hat{\mathbf{A}}_{\infty}(\mathbf{r}, \omega))$$

$$\langle \mathbf{S}_{\infty} \rangle = \frac{1}{2Z_0} \omega^2 |\mathbf{r}_0 \times \hat{\mathbf{A}}_{\infty}(\mathbf{r}, \omega)|^2 \mathbf{r}_0$$


Farfield has a planewave-like geometry



Field in Radiation Zone – General Case TD

$$\mathbf{A}_\infty(\mathbf{r}, t) \approx \frac{\mu}{4\pi r} \int_{V'} \mathbf{J}\left(\mathbf{r}', t - \frac{r}{c_0} + \frac{\mathbf{r}_0 \cdot \mathbf{r}'}{c_0}\right) dV'$$

$$\mathbf{E}_\infty \approx -Z_0(\mathbf{r}_0 \times \mathbf{H}_\infty) \quad \mathbf{H}_\infty(\mathbf{r}, t) \approx -\frac{1}{Z_0} \mathbf{r}_0 \times \dot{\mathbf{A}}_\infty(\mathbf{r}, t)$$


$$\mathbf{E}_\infty(\mathbf{r}, t) \approx \mathbf{r}_0 \times (\mathbf{r}_0 \times \dot{\mathbf{A}}_\infty(\mathbf{r}, t))$$

$$\mathbf{S}_\infty \approx \frac{1}{Z_0} |\mathbf{r}_0 \times \dot{\mathbf{A}}_\infty(\mathbf{r}, t)|^2 \mathbf{r}_0$$

Farfield has a planewave-like geometry



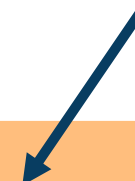
Radiation Zone = Rays

$$\hat{\mathbf{A}}_{\infty}(\mathbf{r}, \omega) \approx \frac{\mu_0}{4\pi} \frac{e^{-jk_0 r}}{r} \int_{V'} \hat{\mathbf{J}}(\mathbf{r}', \omega) e^{jk_0 \mathbf{r}_0 \cdot \mathbf{r}'} dV'$$



4D Fourier('s) transform

$$\hat{\mathbf{J}}(k_x, k_y, k_z, \omega) = \mathcal{F}_{x,y,z,t} \{ \mathbf{J}(x, y, z, t) \}$$



$$\mathbf{A}_{\infty}(\mathbf{r}, t) \approx \mathcal{F}_{\omega}^{-1} \left\{ \frac{\mu_0}{4\pi} \hat{\mathbf{J}}(-k_0 \mathbf{r}_0, \omega) \frac{e^{-jk_0 r}}{r} \right\}$$

Radiation diagram is formed by Fourier('s) transform of sources



Angular Spectrum Representation (Sources)

4D Fourier('s) transform

$$\hat{\mathbf{J}}(k_x, k_y, k_z, \omega) = \mathcal{F}_{x,y,z,t} \{ \mathbf{J}(x, y, z, t) \}$$

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2}$$

$$\text{Im}[k_z] < 0$$

$$\hat{\mathbf{G}} = \frac{\hat{\mathbf{J}}(k_x, k_y, \mp k_z, \omega)}{2k_z} e^{\mp jk_z z}$$

Valid only outside the source region

$$\mathbf{H} \left(x, y, \begin{matrix} z > \max(z') \\ z < \min(z') \end{matrix}, t \right) = \mathcal{F}_{k_x, k_y, \omega}^{-1} \left\{ [k_x, k_y, \mp k_z] \times \hat{\mathbf{G}} \right\}$$

$$\mathbf{E} \left(x, y, \begin{matrix} z > \max(z') \\ z < \min(z') \end{matrix}, t \right) = \mathcal{F}_{k_x, k_y, \omega}^{-1} \left\{ \frac{Z_0}{k} [k_x, k_y, \mp k_z] \times [k_x, k_y, \mp k_z] \times \hat{\mathbf{G}} \right\}$$

General solution to free-space Maxwell's equations

Angular Spectrum in Radiation Zone

$$\mathbf{F}(x, y, z > 0, \omega) = \mathcal{F}_{k_x, k_y}^{-1} \left\{ \hat{\mathbf{G}}(k_x, k_y) e^{-jk_z z} \right\}$$

$$kr \rightarrow \infty \quad \Downarrow \quad \begin{aligned} k_z &= \sqrt{k_0^2 - k_x^2 - k_y^2} \\ \text{Im}[k_z] &< 0 \end{aligned} \quad \mathbf{r}_0 = \frac{[x, y, z]}{r}$$

$$\mathbf{F}_\infty(\mathbf{r}, \omega) = \frac{1}{(2\pi)^2} \int_{k^2 > k_x^2 + k_y^2} \hat{\mathbf{G}}(k_x, k_y) e^{jk_r \left(\frac{k_x}{k} r_{0x} + \frac{k_y}{k} r_{0y} - \frac{k_z}{k} r_{0z} \right)} dk_x dk_y$$

$$kr \rightarrow \infty \quad \Downarrow \quad \text{Stationary phase method}$$

$$\mathbf{F}_\infty(\mathbf{r}, \omega) = \frac{jk_r r_{0z}}{2\pi} \hat{\mathbf{G}}(-kr_{0x}, -kr_{0y}) \frac{e^{-jk_r r}}{r}$$

Farfield is made of propagating planewaves



Angular Spectrum in Radiation Zone

4D Fourier('s) transform

$$\hat{\mathbf{J}}(k_x, k_y, k_z, \omega) = \mathcal{F}_{x,y,z,t} \{ \mathbf{J}(x, y, z, t) \}$$

$$\mathbf{H}_\infty(\mathbf{r}, t) \approx \mathcal{F}_\omega^{-1} \left\{ -\frac{j k}{4\pi} \mathbf{r}_0 \times \hat{\mathbf{J}}(-k_0 \mathbf{r}_0, \omega) \frac{e^{-j k_0 r}}{r} \right\}$$

$$\mathbf{E}_\infty(\mathbf{r}, t) \approx \mathcal{F}_\omega^{-1} \left\{ \frac{j k Z_0}{4\pi} \mathbf{r}_0 \times \mathbf{r}_0 \times \hat{\mathbf{J}}(-k_0 \mathbf{r}_0, \omega) \frac{e^{-j k_0 r}}{r} \right\}$$

$$\mathbf{r}_0 = \frac{[x, y, z]}{r}$$

Farfield is made of propagating planewaves



Planar Material Boundary

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2}$$

Incident wave

Reflected (1 → 1) / Transmitted (2 → 1) wave

$$\begin{aligned}
 \mathbf{H}(z < 0) &= \mathcal{F}_{k_x, k_y}^{-1} \left\{ \mathbf{H}_1^+(k_x, k_y, \omega) e^{-jk_{z1}z} + \mathbf{H}_1^-(k_x, k_y, \omega) e^{jk_{z1}z} \right\} \\
 \mathbf{E}(z < 0) &= \mathcal{F}_{k_x, k_y}^{-1} \left\{ Z_1 \frac{[k_x, k_y, -k_{z1}] \times \mathbf{H}_1^+(k_x, k_y, \omega)}{k_1} e^{-jk_{z1}z} + Z_1 \frac{[k_x, k_y, k_{z1}] \times \mathbf{H}_1^-(k_x, k_y, \omega)}{k_1} e^{jk_{z1}z} \right\} \\
 \mathbf{H}(z > 0) &= \mathcal{F}_{k_x, k_y}^{-1} \left\{ \mathbf{H}_2^+(k_x, k_y, \omega) e^{-jk_{z2}z} + \mathbf{H}_2^-(k_x, k_y, \omega) e^{jk_{z2}z} \right\} \\
 \mathbf{E}(z > 0) &= \mathcal{F}_{k_x, k_y}^{-1} \left\{ Z_2 \frac{[k_x, k_y, -k_{z2}] \times \mathbf{H}_2^+(k_x, k_y, \omega)}{k_2} e^{-jk_{z2}z} + Z_2 \frac{[k_x, k_y, k_{z2}] \times \mathbf{H}_2^-(k_x, k_y, \omega)}{k_2} e^{jk_{z2}z} \right\}
 \end{aligned}$$

Boundary is at $z = 0$

Reflected (2 → 2) / Transmitted (1 → 2) wave

Incident wave



Field is composed of incident, reflected and transmitted waves

Planar Material Boundary – Boundary Conditions

$$\begin{bmatrix} k_x \\ k_y \\ \mp k_{z1} \end{bmatrix} \cdot \mathbf{H}_1^\pm = 0$$

$$\begin{bmatrix} k_x \\ k_y \\ \mp k_{z2} \end{bmatrix} \cdot \mathbf{H}_2^\pm = 0$$

$$\mathbf{z}_0 \times \mathbf{H}_1^+ + \mathbf{z}_0 \times \mathbf{H}_1^- = \mathbf{z}_0 \times \mathbf{H}_2^+ + \mathbf{z}_0 \times \mathbf{H}_2^-$$

$$Z_1 \frac{\mathbf{z}_0 \times \begin{bmatrix} k_x \\ k_y \\ -k_{z1} \end{bmatrix} \times \mathbf{H}_1^+}{k_1} + Z_1 \frac{\mathbf{z}_0 \times \begin{bmatrix} k_x \\ k_y \\ k_{z1} \end{bmatrix} \times \mathbf{H}_1^-}{k_1} =$$

$$Z_2 \frac{\mathbf{z}_0 \times \begin{bmatrix} k_x \\ k_y \\ -k_{z2} \end{bmatrix} \times \mathbf{H}_2^+}{k_2} + Z_2 \frac{\mathbf{z}_0 \times \begin{bmatrix} k_x \\ k_y \\ k_{z2} \end{bmatrix} \times \mathbf{H}_2^-}{k_2}$$

k_x, k_y are equal on both sides

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2}$$

$$\text{Im}[k_z] < 0$$



Relations valid for both propagative and evanescent waves

Perpendicular Incidence – Matrix Form

$$k_x = k_y = 0$$

Transmission matrix
(multilayer cascade)

$$[S] = \frac{1}{T_{22}} \begin{bmatrix} -T_{21} & 1 \\ \det T & T_{12} \end{bmatrix}$$

$$[T] = \frac{1}{S_{12}} \begin{bmatrix} -\det S & S_{22} \\ -S_{11} & 1 \end{bmatrix}$$

$$\begin{bmatrix} E_2^+ \\ E_2^- \end{bmatrix} = [T] \begin{bmatrix} E_1^+ \\ E_1^- \end{bmatrix}$$

$$[T] = \frac{1}{2Z_1} \begin{bmatrix} Z_1 + Z_2 & Z_1 - Z_2 \\ Z_1 - Z_2 & Z_1 + Z_2 \end{bmatrix}$$

$$\begin{bmatrix} E_1^- \\ E_2^+ \end{bmatrix} = [S] \begin{bmatrix} E_1^+ \\ E_2^- \end{bmatrix}$$

$$[S] = \frac{1}{Z_2 + Z_1} \begin{bmatrix} Z_2 - Z_1 & 2Z_1 \\ 2Z_2 & Z_1 - Z_2 \end{bmatrix}$$

Scattering matrix
(experiments)

Matrices with the use across the electrical engineering



Perpendicular Incidence – Interesting Cases

Wavelength inside the slab

Transparent dielectric layer

$$d = \frac{\lambda_2}{2} N$$



$$T = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, S = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Bragg's mirror
(dielectric mirror)

$$k_0 (n_1 d_1 + n_2 d_2) = N\pi$$

Alternating dielectric layers

Technically important special cases



Oblique Incidence – TM / TE Case

$$k_x = k_x$$
$$k_y = 0$$

Snell's law of refraction

$$k_1 \sin \alpha_{\text{inc}} = k_2 \sin \alpha_{\text{trans}}$$

$$k_1 \sin \alpha_{\text{inc}} = k_1 \sin \alpha_{\text{refl}}$$

The angle of incidence equals to the angle of reflection

Snell's law is a property of k-vectors



Oblique Incidence – TM Case

$$\begin{aligned}k_y &= 0 \\ H_x &= 0 \\ H_z &= 0\end{aligned}$$

$$\begin{aligned}R_{1 \rightarrow 1}^{\text{TM}} &= \frac{E_{1x}^-}{E_{1x}^+} = \frac{\sqrt{1 - \frac{k_x^2}{k_2^2}} Z_2 - \sqrt{1 - \frac{k_x^2}{k_1^2}} Z_1}{\sqrt{1 - \frac{k_x^2}{k_2^2}} Z_2 + \sqrt{1 - \frac{k_x^2}{k_1^2}} Z_1} \\ T_{1 \rightarrow 2}^{\text{TM}} &= \frac{E_{2x}^+}{E_{1x}^+} = \frac{2Z_2 \sqrt{1 - \frac{k_x^2}{k_2^2}}}{\sqrt{1 - \frac{k_x^2}{k_2^2}} Z_2 + \sqrt{1 - \frac{k_x^2}{k_1^2}} Z_1}\end{aligned}$$

Generalization of reflection and transmission to oblique incidence



Oblique Incidence – TM Case

$$\begin{aligned}k_y &= 0 \\ H_x &= 0 \\ H_z &= 0\end{aligned}$$

Brewster's angle

$$R_{1 \rightarrow 1}^{\text{TM}} = 0 \quad \Rightarrow \quad \frac{k_x}{k_1} = \sqrt{\frac{\epsilon_2 (\mu_2 \epsilon_1 - \mu_1 \epsilon_2)}{\mu_1 (\epsilon_1^2 - \epsilon_2^2)}}$$

Vanishing reflection on a boundary

Simplification for pure dielectrics

$$\frac{k_x}{k_1} = \left(1 + \frac{\epsilon_1}{\epsilon_2}\right)^{-\frac{1}{2}}$$

Can be used for polarizing unpolarized light beams



Oblique Incidence – TE Case

$$\begin{aligned}k_y &= 0 \\ E_x &= 0 \\ E_z &= 0\end{aligned}$$

$$\begin{aligned}R_{1 \rightarrow 1}^{\text{TE}} &= \frac{E_{1y}^-}{E_{1y}^+} = \frac{\sqrt{1 - \frac{k_x^2}{k_1^2} Z_2} - \sqrt{1 - \frac{k_x^2}{k_2^2} Z_1}}{\sqrt{1 - \frac{k_x^2}{k_1^2} Z_2} + \sqrt{1 - \frac{k_x^2}{k_2^2} Z_1}} \\ T_{1 \rightarrow 2}^{\text{TE}} &= \frac{E_{2y}^+}{E_{1y}^+} = \frac{2Z_2 \sqrt{1 - \frac{k_x^2}{k_1^2}}}{\sqrt{1 - \frac{k_x^2}{k_1^2} Z_2} + \sqrt{1 - \frac{k_x^2}{k_2^2} Z_1}}\end{aligned}$$

Generalization of reflection and transmission to oblique incidence



Oblique Incidence – TE Case

$$k_y = 0$$
$$E_x = 0$$
$$E_z = 0$$

Brewster's angle

$$R_{1 \rightarrow 1}^{\text{TE}} = 0 \quad \longrightarrow \quad \frac{k_x}{k_1} = \sqrt{\frac{\mu_2 (\varepsilon_2 \mu_1 - \varepsilon_1 \mu_2)}{\varepsilon_1 (\mu_1^2 - \mu_2^2)}}$$

Vanishing reflection on a boundary

Simplification for pure magnetics

$$\frac{k_x}{k_1} = \left(1 + \frac{\mu_1}{\mu_2}\right)^{-\frac{1}{2}}$$

Unrealistic scenario for natural materials



Oblique Incidence – Total Reflection

$$\frac{k_x}{k_1} > \frac{k_2}{k_1} = \frac{n_2}{n_1} < 1$$



$$\left| R_{1 \rightarrow 1}^{\text{TM}} \right| = \left| R_{1 \rightarrow 1}^{\text{TE}} \right| = 0$$

Valid for both, the TM and the TE case





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