Electromagnetic Field Theory 2

(fundamental relations and definitions)

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Maxwell('s)-Lorentz('s) Equations

Equations of motion for fields

$$\nabla \times \boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{J}(\boldsymbol{r},t) + \frac{\partial \boldsymbol{D}(\boldsymbol{r},t)}{\partial t}$$

$$\nabla \times \boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t}$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r},t) = 0$$

$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r},t) = \rho(\boldsymbol{r},t)$$

Equation of motion for particles

$$\boldsymbol{f}(\boldsymbol{r},t) = \rho(\boldsymbol{r},t)\boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{J}(\boldsymbol{r},t) \times \boldsymbol{B}(\boldsymbol{r},t)$$

Interaction with materials

$$\boldsymbol{D}(\boldsymbol{r},t) = \varepsilon_0 \boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{P}(\boldsymbol{r},t)$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \mu_0 \left(\boldsymbol{H}(\boldsymbol{r},t) + \boldsymbol{M}(\boldsymbol{r},t)\right)$$

Absolute majority of things happening around us is described by these equations

Boundary Conditions

$$\boldsymbol{n}(\boldsymbol{r}) \times \left[\boldsymbol{E}_1(\boldsymbol{r},t) - \boldsymbol{E}_2(\boldsymbol{r},t)\right] = 0$$

$$\boldsymbol{n}(\boldsymbol{r}) \times \left[\boldsymbol{H}_1(\boldsymbol{r},t) - \boldsymbol{H}_2(\boldsymbol{r},t)\right] = \boldsymbol{K}(\boldsymbol{r},t)$$

$$\boldsymbol{n}(\boldsymbol{r}) \cdot \left[\boldsymbol{B}_1(\boldsymbol{r},t) - \boldsymbol{B}_2(\boldsymbol{r},t)\right] = 0$$

$$\boldsymbol{n}(\boldsymbol{r}) \cdot \left[\boldsymbol{D}_1(\boldsymbol{r},t) - \boldsymbol{D}_2(\boldsymbol{r},t)\right] = \sigma(\boldsymbol{r},t)$$

Normal pointing to region (1)

Electromagnetic Potentials

Lorentz(‘s) calibration

$$\nabla \cdot \boldsymbol{A}\left(\boldsymbol{r},t\right) = -\sigma\mu\varphi\left(\boldsymbol{r},t\right) - \varepsilon\mu\frac{\partial\varphi\left(\boldsymbol{r},t\right)}{\partial t}$$

$$\boldsymbol{B}\left(\boldsymbol{r},t\right) = \nabla \times \boldsymbol{A}\left(\boldsymbol{r},t\right)$$

$$\boldsymbol{E}\left(\boldsymbol{r},t\right) = -\nabla\varphi\left(\boldsymbol{r},t\right) - \frac{\partial\boldsymbol{A}\left(\boldsymbol{r},t\right)}{\partial t}$$

Wave Equation

$$\Delta \boldsymbol{A}(\boldsymbol{r},t) - \sigma\mu \frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} - \varepsilon\mu \frac{\partial^2 \boldsymbol{A}(\boldsymbol{r},t)}{\partial t^2} = -\mu \boldsymbol{J}_{\text{source}}(\boldsymbol{r},t)$$

Material parameters are assumed independent of coordinates

Poynting('s)-Umov('s) Theorem

Power passing the bounding envelope

Energy storage

$$-\int_V \boldsymbol{E}\cdot\boldsymbol{J}_{\mathrm{source}}\,\mathrm{d}V = \oint_S \left(\boldsymbol{E}\times\boldsymbol{H}\right)\cdot\mathrm{d}\boldsymbol{S} + \int_V \sigma\left|\boldsymbol{E}\right|^2\mathrm{d}V + \frac{1}{2}\frac{\partial}{\partial t}\int_V\left(\varepsilon\left|\boldsymbol{E}\right|^2 + \mu\left|\boldsymbol{H}\right|^2\right)\mathrm{d}V$$

Power supplied by sources

Heat losses

Energy balance in an electromagnetic system

Frequency Domain

$\boldsymbol{F}\left(\boldsymbol{r},t\right)\in\mathbb{R}$

$\hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right)\in\mathbb{C}$

$$\boldsymbol{F}\left(\boldsymbol{r},t\right)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right)\mathrm{e}^{\mathrm{j}\omega t}d\omega \quad \Longleftrightarrow \quad \hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right)=\int\limits_{-\infty}^{\infty}\boldsymbol{F}\left(\boldsymbol{r},t\right)\mathrm{e}^{-\mathrm{j}\omega t}dt$$

Time derivatives reduce to algebraic multiplication

$$\frac{\partial\boldsymbol{F}\left(\boldsymbol{r},t\right)}{\partial t}\leftrightarrow\mathrm{j}\omega\hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right)$$

$$\frac{\partial\boldsymbol{F}\left(\boldsymbol{r},t\right)}{\partial r_{\xi}}\leftrightarrow\frac{\partial\hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right)}{\partial r_{\xi}}$$

Spatial derivatives are untouched

Frequency domain helps us to remove explicit time derivatives

Phasors

$$\hat{\boldsymbol{F}}\left(\boldsymbol{r},-\omega\right)=\hat{\boldsymbol{F}}^{*}\left(\boldsymbol{r},\omega\right) \quad \Rightarrow \quad \boldsymbol{F}\left(\boldsymbol{r},t\right)=\frac{1}{\pi}\int\limits_{0}^{\infty}\mathrm{Re}\left[\hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right)\mathrm{e}^{\mathrm{j}\omega t}\right]d\omega$$

Reduced frequency domain representation

Maxwell('s) Equations – Frequency Domain

$$\nabla \times \hat{\boldsymbol{H}}\left(\boldsymbol{r},\omega\right) = \hat{\boldsymbol{J}}\left(\boldsymbol{r},\omega\right) + \mathrm{j}\omega\varepsilon\hat{\boldsymbol{E}}\left(\boldsymbol{r},\omega\right)$$

$$\nabla \times \hat{\boldsymbol{E}}\left(\boldsymbol{r},\omega\right) = -\mathrm{j}\omega\mu\hat{\boldsymbol{H}}\left(\boldsymbol{r},\omega\right)$$

$$\nabla \cdot \hat{\boldsymbol{H}}\left(\boldsymbol{r},\omega\right) = 0$$

$$\nabla \cdot \hat{\boldsymbol{E}}\left(\boldsymbol{r},\omega\right) = \frac{\hat{\rho}\left(\boldsymbol{r},\omega\right)}{\varepsilon}$$

We assume linearity of material relations

Wave Equation – Frequency Domain

$$\Delta \hat{\boldsymbol{A}}\left(\boldsymbol{r},\omega\right) - \mathrm{j}\omega\mu\left(\sigma + \mathrm{j}\omega\varepsilon\right)\hat{\boldsymbol{A}}\left(\boldsymbol{r},\omega\right) = -\mu\hat{\boldsymbol{J}}_{\mathrm{source}}\left(\boldsymbol{r},\omega\right)$$

Helmholtz('s) equation

Heat Balance in Time-Harmonic Steady State

Valid for general periodic steady state

Cycle mean

$$-\int_V \left\langle \boldsymbol{E}\cdot\boldsymbol{J}_{\text{source}}\right\rangle \mathrm{d}V = \oint_S \left\langle \boldsymbol{E}\times\boldsymbol{H}\right\rangle\cdot\mathrm{d}\boldsymbol{S} + \int_V \left\langle \sigma\left|\boldsymbol{E}\right|^2\right\rangle \mathrm{d}V$$

$$-\frac{1}{2}\int_V \mathrm{Re}\left[\hat{\boldsymbol{E}}\cdot\hat{\boldsymbol{J}}^*_{\text{source}}\right]\mathrm{d}V = \frac{1}{2}\oint_S \mathrm{Re}\left[\hat{\boldsymbol{E}}\times\hat{\boldsymbol{H}}^*\right]\cdot\mathrm{d}\boldsymbol{S} + \frac{1}{2}\int_V \sigma\left|\hat{\boldsymbol{E}}\right|^2\mathrm{d}V$$

Valid for time-harmonic steady state

Plane Wave

$$\hat{\boldsymbol{E}}\left(\boldsymbol{r},\omega\right) = \boldsymbol{E}_0\left(\omega\right)\mathrm{e}^{-\mathrm{j}k\boldsymbol{n}\cdot\boldsymbol{r}}$$

$$\hat{\boldsymbol{H}}\left(\boldsymbol{r},\omega\right) = \frac{k}{\omega\mu}\left[\boldsymbol{n}\times\boldsymbol{E}_0\left(\omega\right)\right]\mathrm{e}^{-\mathrm{j}k\boldsymbol{n}\cdot\boldsymbol{r}}$$

$$\boldsymbol{n}\cdot\boldsymbol{E}_0\left(\omega\right) = 0$$

$$\boldsymbol{n}\cdot\boldsymbol{H}_0\left(\omega\right) = 0$$

$$k^2 = -\mathrm{j}\omega\mu\left(\sigma + \mathrm{j}\omega\varepsilon\right)$$

Unitary vector representing the direction of propagation

Electric and magnetic fields are mutually orthogonal

Electric and magnetic fields are orthogonal to propagation direction

Wave-number

The simplest wave solution of Maxwell(‘s) equations

Plane Wave Characteristics

$$k = \sqrt{-\mathrm{j}\omega\mu\left(\sigma + \mathrm{j}\omega\varepsilon\right)}$$

$$\mathrm{Re}\left[k\right] > 0;\ \mathrm{Im}\left[k\right] < 0$$

$$\lambda = \frac{2\pi}{\mathrm{Re}\left[k\right]}$$

$$v_{\mathrm{f}} = \frac{\omega}{\mathrm{Re}\left[k\right]}$$

$$Z = \frac{\omega\mu}{k}$$

$$\delta = -\frac{1}{\mathrm{Im}\left[k\right]}$$

General isotropic material

Vacuum

$$k = \frac{\omega}{c_0}$$

$$\mathrm{Re}\left[k\right] > 0;\ \mathrm{Im}\left[k\right] = 0$$

$$\lambda = \frac{c_0}{f}$$

$$v_{\mathrm{f}} = c_0$$

$$Z = c_0\mu_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377\ \Omega$$

$$\delta \to \infty$$

Cycle Mean Power Density of a Plane Wave

Power propagation coincides with phase propagation

$$\left\langle \boldsymbol{E}\left(\boldsymbol{r},t\right)\times\boldsymbol{H}\left(\boldsymbol{r},t\right)\right\rangle = \frac{1}{2}\frac{\mathrm{Re}\left[k\right]}{\omega\mu}\left|\boldsymbol{E}_0\left(\omega\right)\right|^2 \mathrm{e}^{2\,\mathrm{Im}\left[k\right]\boldsymbol{n}\cdot\boldsymbol{r}}\boldsymbol{n}$$

Source Free Maxwell('s) Equations in Free Space

$$\nabla \times \boldsymbol{H}\left(\boldsymbol{r},t\right) = \sigma\left(t\right) * \boldsymbol{E}\left(\boldsymbol{r},t\right) + \varepsilon\left(t\right) * \frac{\partial \boldsymbol{E}\left(\boldsymbol{r},t\right)}{\partial t}$$

$$\nabla \times \boldsymbol{E}\left(\boldsymbol{r},t\right) = -\mu\left(t\right) * \frac{\partial \boldsymbol{H}\left(\boldsymbol{r},t\right)}{\partial t}$$

$$\nabla \cdot \boldsymbol{H}\left(\boldsymbol{r},t\right) = 0$$

$$\nabla \cdot \boldsymbol{E}\left(\boldsymbol{r},t\right) = 0$$

$$\left|\boldsymbol{k}\right|^2 = k^2 = -\mathrm{j}\omega\hat{\mu}\left(\omega\right)\left(\hat{\sigma}\left(\omega\right) + \mathrm{j}\omega\hat{\varepsilon}\left(\omega\right)\right)$$

$$\hat{\boldsymbol{E}}\left(\boldsymbol{k},\omega\right) = \frac{\boldsymbol{k}}{\omega\hat{\varepsilon}\left(\omega\right) - \mathrm{j}\hat{\sigma}\left(\omega\right)} \times \hat{\boldsymbol{H}}\left(\boldsymbol{k},\omega\right)$$

$$\hat{\boldsymbol{H}}\left(\boldsymbol{k},\omega\right) = -\frac{\boldsymbol{k}}{\omega\hat{\mu}\left(\omega\right)} \times \hat{\boldsymbol{E}}\left(\boldsymbol{k},\omega\right)$$

$$\boldsymbol{k} \cdot \hat{\boldsymbol{E}}\left(\boldsymbol{k},\omega\right) = 0$$

$$\boldsymbol{k} \cdot \hat{\boldsymbol{H}}\left(\boldsymbol{k},\omega\right) = 0$$

$$\boldsymbol{F}\left(\boldsymbol{r},t\right) = \frac{1}{\left(2\pi\right)^4} \int\limits_{\boldsymbol{k},t} \hat{\boldsymbol{F}}\left(\boldsymbol{k},\omega\right) \mathrm{e}^{\mathrm{j}\left(\boldsymbol{k}\cdot\boldsymbol{r}+\omega t\right)} \mathrm{d}\boldsymbol{k}\mathrm{d}\omega$$

Fourier's transform leads to simple algebraic equations

Spatial Wave Packet

$$\left|\boldsymbol{k}\right|^2 = k^2 = -\mathrm{j}\omega\hat{\mu}\left(\omega\right)\left(\hat{\sigma}\left(\omega\right) + \mathrm{j}\omega\hat{\varepsilon}\left(\omega\right)\right) \quad \Rightarrow \quad \omega = \omega\left(\left|\boldsymbol{k}\right|\right)$$

This can be electric or magnetic intensity

$$\boldsymbol{F}\left(\boldsymbol{r},t\right) = \frac{1}{\left(2\pi\right)^3}\int\limits_{\boldsymbol{k}} \hat{\boldsymbol{F}}_0\left(\boldsymbol{k}\right)\mathrm{e}^{\mathrm{j}\left(\boldsymbol{k}\cdot\boldsymbol{r}+\omega\left(\left|\mathbf{k}\right|\right)t\right)}\mathrm{d}\boldsymbol{k}$$

$$\boldsymbol{k}\cdot\hat{\boldsymbol{F}}_0\left(\boldsymbol{k}\right) = 0$$

General solution to free-space Maxwell's equations

Spatial Wave Packet in Vacuum

$$\omega\left(\left|\boldsymbol{k}\right|\right) = \pm c_0 \left|\boldsymbol{k}\right|$$

$$\boldsymbol{k} \cdot \hat{\boldsymbol{F}}_0^+\left(\boldsymbol{k}\right) = \boldsymbol{k} \cdot \hat{\boldsymbol{F}}_0^-\left(\boldsymbol{k}\right) = 0$$

$$\hat{\boldsymbol{F}}_0^-\left(\boldsymbol{k}\right) = \left[\hat{\boldsymbol{F}}_0^+\left(-\boldsymbol{k}\right)\right]^*$$

$$\hat{\boldsymbol{F}}_0^+\left(\boldsymbol{k}\right) = \left[\hat{\boldsymbol{F}}_0^-\left(-\boldsymbol{k}\right)\right]^*$$

$$\boldsymbol{F}\left(\boldsymbol{r},t\right) = \frac{1}{\left(2\pi\right)^3}\int_{\boldsymbol{k}} \mathrm{e}^{\mathrm{j}\boldsymbol{k}\cdot\boldsymbol{r}}\left[\hat{\boldsymbol{F}}_0^+\left(\boldsymbol{k}\right)\mathrm{e}^{\mathrm{j}c_0 t\left|\boldsymbol{k}\right|} + \hat{\boldsymbol{F}}_0^-\left(\boldsymbol{k}\right)\mathrm{e}^{-\mathrm{j}c_0 t\left|\boldsymbol{k}\right|}\right]\mathrm{d}\boldsymbol{k}$$

$$\hat{\boldsymbol{F}}_0^+\left(\boldsymbol{k}\right) = \frac{1}{2}\int_{\boldsymbol{r}}\left[\boldsymbol{F}\left(\boldsymbol{r},0\right) + \frac{1}{\mathrm{j}c_0\left|\boldsymbol{k}\right|}\left.\frac{\partial \mathbf{F}\left(\boldsymbol{r},t\right)}{\partial t}\right|_{t=0}\right]\mathrm{e}^{-\mathrm{j}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{r}$$

$$\hat{\boldsymbol{F}}_0^-\left(\boldsymbol{k}\right) = \frac{1}{2}\int_{\boldsymbol{r}}\left[\boldsymbol{F}\left(\boldsymbol{r},0\right) - \frac{1}{\mathrm{j}c_0\left|\boldsymbol{k}\right|}\left.\frac{\partial \mathbf{F}\left(\boldsymbol{r},t\right)}{\partial t}\right|_{t=0}\right]\mathrm{e}^{-\mathrm{j}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{r}$$

The field is uniquely given by initial conditions

Spatial Wave Packet in Vacuum

$$\omega\left(\left|\boldsymbol{k}\right|\right) = \pm c_0 \left|\boldsymbol{k}\right|$$

$$\boldsymbol{E}\left(\boldsymbol{r},t\right) = \frac{1}{\left(2\pi\right)^3}\int\limits_{\mathbf{k}} \mathrm{e}^{\mathrm{j}\boldsymbol{k}\cdot\boldsymbol{r}}\left[\hat{\boldsymbol{E}}^{+}\left(\boldsymbol{k}\right)\mathrm{e}^{\mathrm{j}c_0 t\left|\boldsymbol{k}\right|} + \hat{\boldsymbol{E}}^{-}\left(\boldsymbol{k}\right)\mathrm{e}^{-\mathrm{j}c_0 t\left|\boldsymbol{k}\right|}\right]\mathrm{d}\boldsymbol{k}$$

$$\boldsymbol{H}\left(\boldsymbol{r},t\right) = -\frac{1}{\left(2\pi\right)^3}\int\limits_{\mathbf{k}} \mathrm{e}^{\mathrm{j}\boldsymbol{k}\cdot\boldsymbol{r}}\frac{\boldsymbol{k}}{Z_0\left|\boldsymbol{k}\right|}\times\left[\hat{\boldsymbol{E}}^{+}\left(\boldsymbol{k}\right)\mathrm{e}^{\mathrm{j}c_0 t\left|\boldsymbol{k}\right|} - \hat{\boldsymbol{E}}^{-}\left(\boldsymbol{k}\right)\mathrm{e}^{-\mathrm{j}c_0 t\left|\boldsymbol{k}\right|}\right]\mathrm{d}\boldsymbol{k}$$

$$\boldsymbol{k}\cdot\hat{\boldsymbol{E}}^{+}\left(\boldsymbol{k}\right) = \boldsymbol{k}\cdot\hat{\boldsymbol{E}}^{-}\left(\boldsymbol{k}\right) = 0$$

Electric and magnetic field are not independent

Vacuum Dispersion

$$\hat{\boldsymbol{E}}^{+}\left(\boldsymbol{k}\right) = \hat{\mathbf{E}}^{-}\left(\mathbf{k}\right) = \left(2\pi\right)^{2} \hat{\boldsymbol{E}}^{+}\left(k_{z}\right)\delta\left(k_{x}\right)\delta\left(k_{y}\right)$$

$$\mathbf{z}_{0} \cdot \hat{\boldsymbol{E}}^{+}\left(k_{z}\right) = \mathbf{z}_{0} \cdot \hat{\boldsymbol{E}}^{-}\left(k_{z}\right) = 0$$

1D problem

Propagation in one direction

no dispersion

$$\boldsymbol{E}\left(z,t\right) = \boldsymbol{E}^{+}\left(z + c_{0}t\right) + \boldsymbol{E}^{-}\left(z - c_{0}t\right)$$

$$\boldsymbol{H}\left(z,t\right) = -\frac{1}{Z_{0}}\mathbf{z}_{0} \times \left[\boldsymbol{E}^{+}\left(z + c_{0}t\right) - \boldsymbol{E}^{-}\left(z - c_{0}t\right)\right]$$

1D waves in vacuum propagate without dispersion

Vacuum Dispersion

In general this term does not represent translation

$$\left(\left[x, y, z\right] \pm c_0 t\right)$$

$$\boldsymbol{E}\left(\boldsymbol{r}, t\right) = \frac{1}{\left(2\pi\right)^3} \int_{\mathrm{k}} \mathrm{e}^{\mathrm{j}\boldsymbol{k}\cdot\boldsymbol{r}} \left[\hat{\boldsymbol{E}}^{+}\left(\boldsymbol{k}\right)\mathrm{e}^{\mathrm{j}c_0 t\left|\boldsymbol{k}\right|} + \hat{\boldsymbol{E}}^{-}\left(\boldsymbol{k}\right)\mathrm{e}^{-\mathrm{j}c_0 t\left|\boldsymbol{k}\right|}\right] \mathrm{d}\boldsymbol{k}$$

Waves propagating in all directions

2D and 3D waves in vacuum always disperse = change shape in time

Angular Spectrum Representation

$$\mathrm{Im}\left[k_z\right] < 0$$

$$\left|\boldsymbol{k}\right|^2 = k^2 = -\mathrm{j}\omega\hat{\mu}\left(\omega\right)\left(\hat{\sigma}\left(\omega\right) + \mathrm{j}\omega\hat{\varepsilon}\left(\omega\right)\right) \Rightarrow k_z = \pm\sqrt{k^2 - k_x^2 - k_y^2}$$

$$\hat{\mathbf{H}}_0\left(k_x, k_y, \omega\right) = -\frac{\mathbf{k}}{Z\left|\mathbf{k}\right|} \times \hat{\mathbf{E}}_0\left(k_x, k_y, \omega\right)$$

$$\hat{\boldsymbol{E}}_0\left(k_x, k_y, \omega\right) = \mathcal{F}_{x,y,t}\left\{\boldsymbol{E}\left(x, y, 0, t\right)\right\}$$

$$\boldsymbol{E}\left(x, y, z < 0, t\right) = \frac{1}{\left(2\pi\right)^3} \int\limits_{k_x, k_y, \omega} \mathrm{e}^{\mathrm{j}\left(k_x x + k_y y + \omega t\right)} \hat{\boldsymbol{E}}_0\left(k_x, k_y, \omega\right) \mathrm{e}^{\mathrm{j}\sqrt{k^2 - k_x^2 - k_y^2}\,z} \mathrm{d}k_x \mathrm{d}k_y \mathrm{d}\omega$$

$$\boldsymbol{E}\left(x, y, z > 0, t\right) = \frac{1}{\left(2\pi\right)^3} \int\limits_{k_x, k_y, \omega} \mathrm{e}^{\mathrm{j}\left(k_x x + k_y y + \omega t\right)} \hat{\boldsymbol{E}}_0\left(k_x, k_y, \omega\right) \mathrm{e}^{-\mathrm{j}\sqrt{k^2 - k_x^2 - k_y^2}\,z} \mathrm{d}k_x \mathrm{d}k_y \mathrm{d}\omega$$

$$\boldsymbol{k} \cdot \hat{\boldsymbol{E}}_0 = 0$$

General solution to free-space Maxwell's equations

Propagating vs Evanescent Waves

$$k_x^2 + k_y^2 < k^2$$

These waves propagate and
can carry information to far distances

$$k_x^2 + k_y^2 > k^2$$

These waves exponentially decay in amplitude
and cannot carry information to far distances

Field picture losses it resolution with distance from the source plane

Paraxial Waves

$$\hat{\boldsymbol{E}}_0\left(k_x, k_y, \omega\right) \Rightarrow k_x^2 + k_y^2 \ll k^2 \Rightarrow \sqrt{k^2 - k_x^2 - k_y^2} \approx k - \frac{1}{2k}\left(k_x^2 + k_y^2\right)$$

$$\boldsymbol{E}\left(x, y, z > 0, t\right) = \frac{1}{\left(2\pi\right)^3} \int\limits_{k_x, k_y, \omega} \mathrm{e}^{\mathrm{j}\left(k_x x + k_y y - kz + \omega t\right)} \hat{\boldsymbol{E}}_0\left(k_x, k_y, \omega\right) \mathrm{e}^{\mathrm{j}\frac{1}{2k}\left(k_x^2 + k_y^2\right) z} \mathrm{d}k_x \mathrm{d}k_y \mathrm{d}\omega$$

$$\boldsymbol{k} \cdot \hat{\boldsymbol{E}}_0 = 0$$

Propagates almost as a planewave

$$\mathbf{z}_0 \cdot \hat{\boldsymbol{E}}_0 \approx 0$$

Gaussian Beam

$$\hat{\boldsymbol{E}}_{0\perp}\left(k_x,k_y,\omega\right)=\mathbf{A}_{0\perp}\pi w_0^2 e^{-\frac{1}{4}w_0^2\left(k_x^2+k_y^2\right)}$$

$$\mathbf{E}_{\perp}\left(x,y,z>0,t\right)=\frac{1}{2\pi}\int_{\omega}\mathbf{A}_{0\perp}\frac{w_0}{w\left(z\right)}\mathrm{e}^{-\frac{x^2+y^2}{w^2\left(z\right)}}\mathrm{e}^{\mathrm{j}\arctan\left(\frac{z}{z_R}\right)}\mathrm{e}^{-\mathrm{j}\frac{x^2+y^2}{w^2\left(z\right)}\left[\frac{z}{z_R}\right]}\mathrm{e}^{\mathrm{j}\omega\left[t-\frac{z}{c_0}n\right]}\mathrm{d}\omega$$

Gaussian profile in amplitude

Paraxial corrections

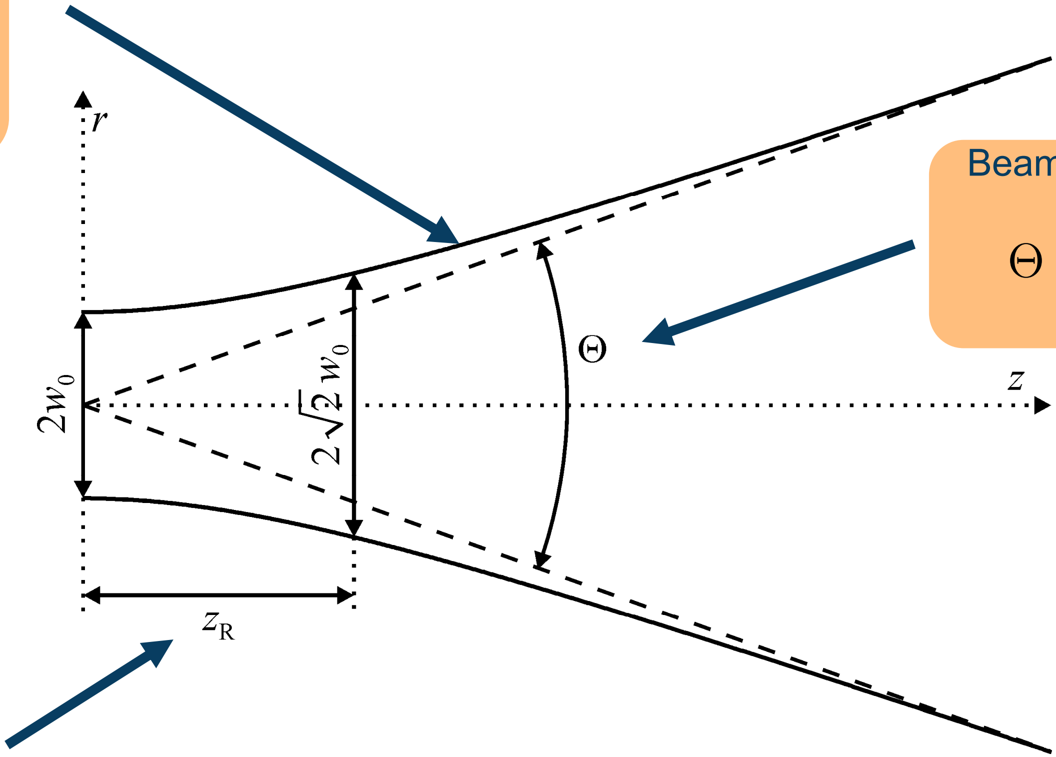
Planewave-like propagation

Approximates radiation of sources large in comparison to wavelength

Gaussian Beam

Half-width of the beam

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2}$$



Beam divergence

$$\Theta = \frac{2\lambda}{\pi w_0}$$

Rayleigh distance

$$z_R = \frac{1}{2} k w_0^2 = \frac{\pi w_0^2}{\lambda}$$

Gaussian Beam – Time-Harmonic Case

$$\langle \boldsymbol{S} \rangle = \frac{1}{2}\mathrm{Re}\left[\hat{\boldsymbol{E}}\left(x,y,z,\omega\right)\times \hat{\boldsymbol{H}}^*\left(x,y,z,\omega\right)\right] = \mathbf{z}_0 S_0 \frac{w_0^2}{w^2\left(z\right)} e^{-\frac{2\rho^2}{w^2\left(z\right)}}$$

86.5 % of power flows through the beam width

Power density at origin

$$w\left(z\right) = w_0\sqrt{1+\left(\frac{z}{z_R}\right)^2}$$

Material Dispersion

Causality requirement

$$\varepsilon\left(\tau\right)=0,\tau<0$$

Stability requirement

$$\varepsilon\left(\tau\right)\to 0,\tau\to\infty$$

$$\boldsymbol{D}\left(\boldsymbol{r},t\right)=\int_{-\infty}^{\infty}\varepsilon\left(\tau\right)\boldsymbol{E}\left(\boldsymbol{r},t-\tau\right)\mathrm{d}\tau$$

$$\boldsymbol{B}\left(\boldsymbol{r},t\right)=\int_{-\infty}^{\infty}\mu\left(\tau\right)\boldsymbol{H}\left(\boldsymbol{r},t-\tau\right)\mathrm{d}\tau$$

$$\boldsymbol{J}\left(\boldsymbol{r},t\right)=\int_{-\infty}^{\infty}\sigma\left(\tau\right)\boldsymbol{E}\left(\boldsymbol{r},t-\tau\right)\mathrm{d}\tau$$

$$\hat{\boldsymbol{D}}\left(\boldsymbol{r},\omega\right)=\hat{\varepsilon}\left(\omega\right)\hat{\boldsymbol{E}}\left(\boldsymbol{r},\omega\right)$$

$$\hat{\boldsymbol{B}}\left(\boldsymbol{r},\omega\right)=\hat{\mu}\left(\omega\right)\hat{\boldsymbol{H}}\left(\boldsymbol{r},\omega\right)$$

$$\hat{\boldsymbol{J}}\left(\boldsymbol{r},\omega\right)=\hat{\sigma}\left(\omega\right)\hat{\boldsymbol{E}}\left(\boldsymbol{r},\omega\right)$$

Even single planewave undergoes time dispersion when materials are present

Lorentz's Dispersion Model

Losses — Resonance — Coupling strength

$$\frac{\partial^2 \boldsymbol{P}(t)}{\partial t^2} + \Gamma \frac{\partial \boldsymbol{P}(t)}{\partial t} + \omega_0^2 \boldsymbol{P}(t) = \varepsilon_0 \omega_\mathrm{p}^2 \boldsymbol{E}(t)$$

$$\varepsilon(\omega) = \varepsilon_0 \left(1 + \frac{\omega_\mathrm{p}^2}{\omega_0^2 - \omega^2 + \mathrm{j}\omega\Gamma}\right)$$

$$\varepsilon(\omega) = \varepsilon_0 \left(1 + \sum_i \frac{\omega_{\mathrm{p},i}^2}{\omega_{0,i}^2 - \omega^2 + \mathrm{j}\omega\Gamma_i}\right)$$

Dispersion model able to describe vast amount of natural materials

Drude‘s Dispersion Model

Special case of
Lorentz‘s dispersion

$$\omega_0 = 0$$

$$\omega_{\mathrm{p}}^2 = \frac{\sigma_0 \Gamma}{\varepsilon_0}$$

Permittivity model

$$\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{\omega_{\mathrm{p}}^2}{\omega(\omega - \mathrm{j}\Gamma)}\right)$$

↔

Conductivity model

$$\sigma(\omega) = \frac{\sigma_0}{1 + \mathrm{j}\dfrac{\omega}{\Gamma}}$$

Collisionless plasma

$$\frac{\Gamma}{\omega} \ll 1 \Rightarrow \varepsilon(\omega) \approx \varepsilon_0 \left(1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2}\right)$$

Dispersion model describing neutral plasma

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