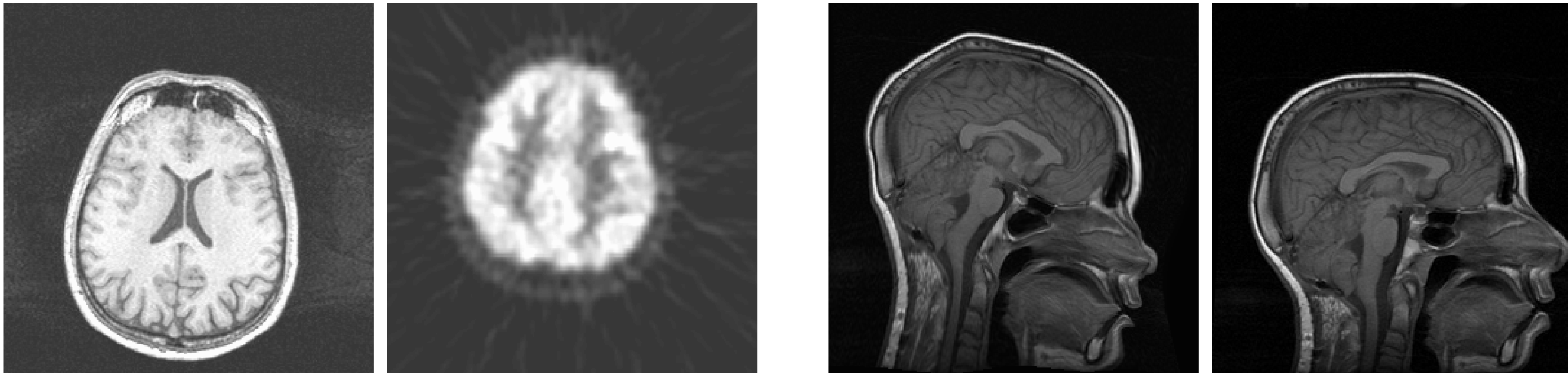
Medical Imaging
Lecture: Multi-modal Image Registration

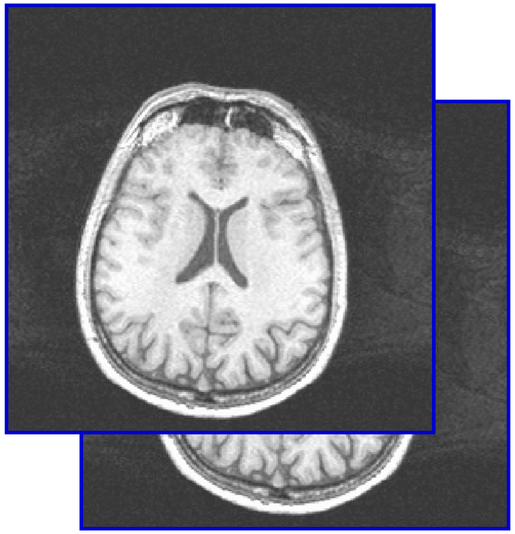
1. Image Registration in Medicine



- Registration – a problem common to many tasks in medical imaging: align two or more images. Images can be two dimensional or three dimensional.
- Aspects of registration:
  - measured object: intra patient – inter individual (e.g. atlas)
  - measuring device: single modality – multi-modal (CT-MRT, MRT-PET,. . .)
  - Transformation: rigid body, affine transformation – smooth but non-linear transformation.

2. Simple case: single patient, single modality

A. Simplest case We start with the simplest variant: register two images of the same patient acquired with the same sensor:



Images: u, u', defined on domains $\mathcal{D}$, $\mathcal{D}'$ respectively, where

- $\mathcal{D}, \mathcal{D}' \subset \mathbb{Z}^2$ $(\mathbb{Z}^3)$
- $u(\boldsymbol{r})$ denotes the image value at pixel (voxel) $\boldsymbol{r}$.

Task (informal): Find the *best* alignment of the images.

Task (formal): Find the rotation matrix R and the translation vector $\boldsymbol{t}$ which minimise the sum of squared differences

$$(R^*, \boldsymbol{t}^*) = \arg\min_{R,\boldsymbol{t}} \frac{1}{|\mathcal{A}|} \sum_{\boldsymbol{r}\in\mathcal{A}} \Big[u(\boldsymbol{r}) - u'(R\boldsymbol{r} + \boldsymbol{t}) \Big]^2$$

where $\mathcal{A}$ denotes the subset of matched voxels $\mathcal{A} = \big\{ \boldsymbol{r} \in D \mid R\boldsymbol{r} + \boldsymbol{t} \in D' \big\}$.

2. Simple case: single patient, single modality

Problems:

- Points $\boldsymbol{r}' = R\boldsymbol{r} + \boldsymbol{t}$ have non-integer coordinates. Image values $u'(\boldsymbol{r}')$?
- Criterion depends on the matching region $\mathcal{A}$.

Algorithms:

Here we consider only translations $\boldsymbol{t}$ (for the sake of simplicity)

- Complete enumeration of all possible (integer valued) translations $\boldsymbol{t}$.
- If: image is a "smooth" function and the translation is small as compared with the characteristic length of variation of image values – approximation by Taylor series

$$u'(\boldsymbol{r} + \boldsymbol{t}) \approx u'(\boldsymbol{r}) + \nabla u'(\boldsymbol{r}) \cdot \boldsymbol{t}$$

The optimisation problem reads

$$\boldsymbol{t}^* = \arg\min_{\boldsymbol{t}} \frac{1}{|\mathcal{A}|} \sum_{\boldsymbol{r} \in \mathcal{A}} \Big[u(\boldsymbol{r}) - u'(\boldsymbol{r}) - \nabla u'(\boldsymbol{r}) \cdot \boldsymbol{t} \Big]^2$$

2. Simple case: single patient, single modality

Let us ignore that $\mathcal{A}$ depends on the translation $\boldsymbol{t}$.

$$F(\boldsymbol{t}) = \sum_{\boldsymbol{r}\in\mathcal{A}} \Big[u(\boldsymbol{r}) - u'(\boldsymbol{r}) - \nabla u'(\boldsymbol{r}) \cdot \boldsymbol{t} \Big]^2 \to \min_{\boldsymbol{t}}$$

$$F(\boldsymbol{t}) = \sum_{\boldsymbol{r}\in\mathcal{A}} \Big[a(\boldsymbol{r}) - \boldsymbol{b}(\boldsymbol{r}) \cdot \boldsymbol{t} \Big]^2 \to \min_{\boldsymbol{t}}$$

Find the minimiser by setting the derivative w.r.t. $\boldsymbol{t}$ to zero

$$\frac{\partial}{\partial \boldsymbol{t}} F(\boldsymbol{t}) = 2 \sum_{\boldsymbol{r}\in\mathcal{A}} \Big[a(\boldsymbol{r}) - \boldsymbol{b}(\boldsymbol{r}) \cdot \boldsymbol{t} \Big] \boldsymbol{b}(\boldsymbol{r}) = 0$$

We get

$$C\,\boldsymbol{t} - \boldsymbol{d} = 0 \quad \Rightarrow \quad \boldsymbol{t}^* = C^{-1}\,\boldsymbol{d}$$

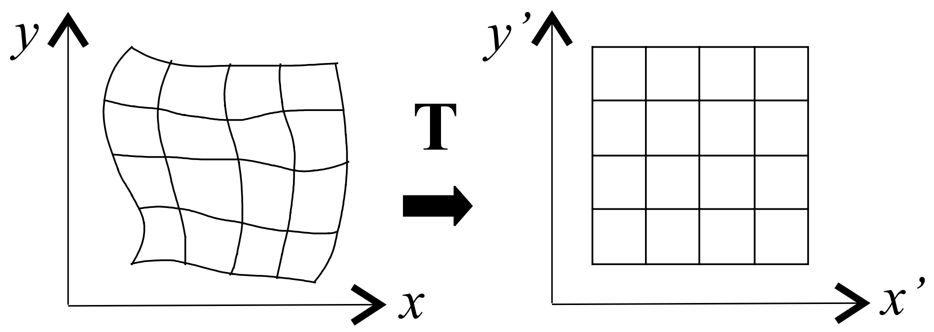
where

$$\boldsymbol{d} = \sum_{\boldsymbol{r}\in\mathcal{A}} a(\boldsymbol{r})\boldsymbol{b}(\boldsymbol{r}) \quad \text{and} \quad C = \sum_{\boldsymbol{r}\in\mathcal{A}} \boldsymbol{b}(\boldsymbol{r}) \otimes \boldsymbol{b}(\boldsymbol{r})$$

3. Transformations and image interpolation

Transformation of pixel (voxel) coordinates $\boldsymbol{r}' = T(\boldsymbol{r})$. In coordinates:

$$x' = T_x(x, y)$$
$$y' = T_y(x, y)$$



- Affine transform in $\mathbb{R}^2$ is given by $\boldsymbol{r}' = A\,\boldsymbol{r} + \boldsymbol{t}$, where: $\boldsymbol{t}$ – translation vector, A – 2x2 matrix with non-zero determinant.
- General non-linear transform in $\mathbb{R}^2$

$$x' = f(x, y) = x + \tilde{f}(x, y)$$
$$y' = h(x, y) = y + \tilde{h}(x, y)$$

where f, h are general non-linear functions s.t. the Jacobian has everywhere non-zero determinant.

3. Transformations and image interpolation

The point $\boldsymbol{r} = (x, y)$ has non-integer coordinates. Which image value $u(\boldsymbol{r})$ should be assigned?

- The point (x, y) is assigned the value of u at the nearest point in the discrete raster.

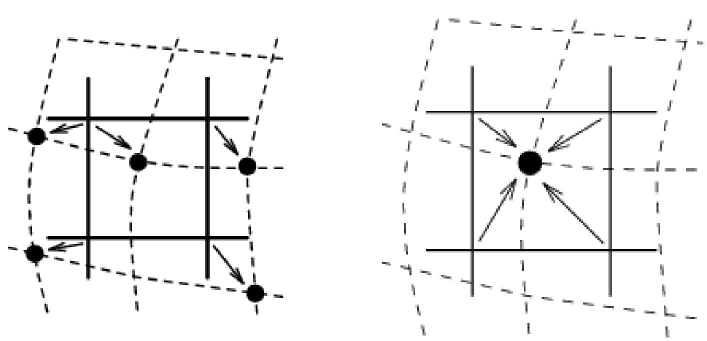
$$u(x,y) = u(\mathrm{round}\,(x),\, \mathrm{round}\,(y))$$

- Linearly combine image values of the 4 neighbouring points

$$u(x,y) = (1-a)\,(1-b)\,u(l,k) + a\,(1-b)\,u(l+1,k) + \\ b\,(1-a)\,u(l,k+1) + a\,b\,u(l+1,k+1)$$

where

$$l = \lfloor x \rfloor, \quad a = x - l\,, \\ k = \lfloor y \rfloor, \quad b = y - k\,.$$



4. Multi-modal image registration

- Images u, v defined on domains D, D' are captured by different sensors.
- Mean square difference error is not applicable in this case.

Idea: consider the **joint histogram** of u and the transformed image v.

- Images well registered: image values of u and v correlated
- Images not well registered: image values of u and v statistically independent

Let $\hat{v}(\boldsymbol{r}) = v\big(T(\boldsymbol{r})\big)$ denote the transformed image v.

Define the histograms

$$p_{u\hat{v}}(k,k') = \frac{1}{|\mathcal{A}|}\Big|\big\{\boldsymbol{r}\in\mathcal{A} \mid u(\boldsymbol{r}) = k,\ \hat{v}(\boldsymbol{r}) = k'\big\}\Big|$$

$$p_u(k) = \frac{1}{|\mathcal{A}|}\Big|\big\{\boldsymbol{r}\in\mathcal{A} \mid u(\boldsymbol{r}) = k\big\}\Big|$$

Mutual information

$$I(u,\hat{v}) = \sum_{k,k'\in G} p_{u\hat{v}}(k,k') \log \frac{p_{u\hat{v}}(k,k')}{p_u(k)p_{\hat{v}}(k')}$$

We expect $I(u,\hat{v})$ to be **maximal** for the true transformation T^*.

4. Multi-modal image registration

Let us understand this:

- Let $p(x)$, $q(x)$, $x = 1, \ldots, n$ be two discrete probability distributions
- Kullback-Leibler divergence

$$D(p,q) = \sum_{x=1}^{n} p(x) \log \frac{p(x)}{q(x)}$$

  has property $D(p,q) \geqslant 0$ and $D(p,q) = 0 \Leftrightarrow p(x) = q(x)\ \forall i$

- In our case: $p \Rightarrow p_{u\hat{v}}(k,k')$ and $q \Rightarrow p_u(k)p_{\hat{v}}(k')$.

4. Multi-modal image registration

Examples:

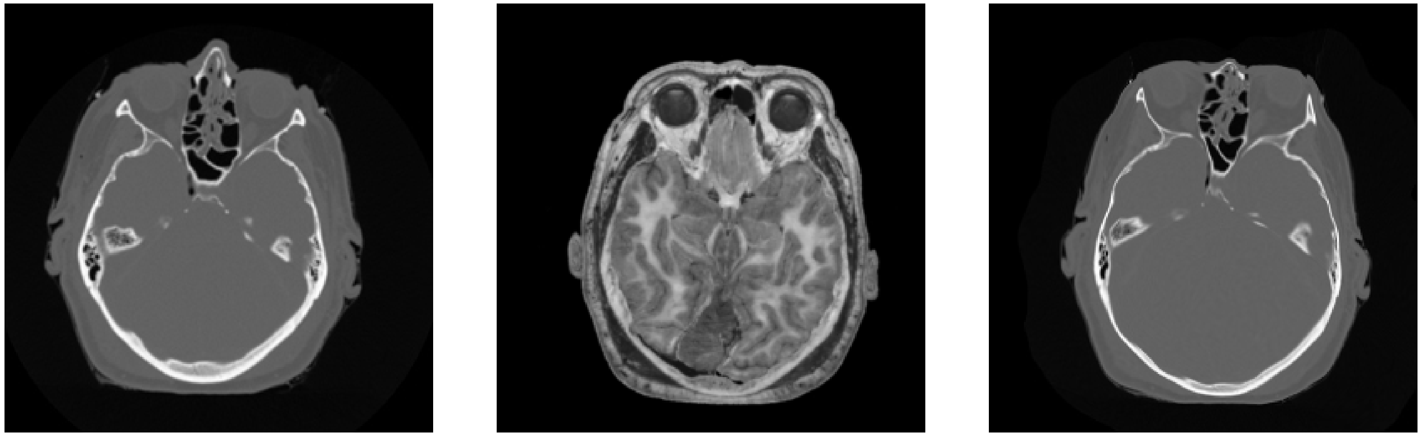
- X, Y are two independent random discrete random variables: $p(x,y) = p(x)p(y)$. Hence

$$I(X,Y) = \sum_{x,y=1}^{n} p(x,y) \log \frac{p(x,y)}{p(x)p(y)} = \sum_{x,y=1}^{n} p(x)p(y) \log 1 = 0$$

- X, Y are both uniform, i.e. $p(x) \equiv p(y) \equiv \frac{1}{n}$, but strongly correlated, e.g. $p(x,y) = \frac{1}{n}\delta_{xy}$. Then

$$I(X,Y) = \sum_{x,y=1}^{n} p(x,y) \log \frac{p(x,y)}{p(x)p(y)} \sum_{x,y=1}^{n} \frac{1}{n}\delta_{xy} \log \frac{n^2}{n}\delta_{xy} = \log n$$

Multi-Modal Registration



source target registered source

(Examples from: S. Periaswamy, H. Farid, Elastic Registration with Partial Data, 2003.)

Discussion:

- Why mutual information?
- Criterion still depends on the matching region $\mathcal{A}$.
- Better results can be obtained if a (probabilistic) anatomical atlas is available.