1 Lecture

1.1 Foundations

1. R is a relation over A and $B \equiv R \subseteq A \times B$.
2. f is a function from A to $B \equiv f \subseteq A \times B \wedge (\forall x \in A\colon \forall y, z \in B\colon (x, y) \in f \wedge (x, z) \in f \Rightarrow y = z)$.
3. $xRy \equiv (x, y) \in R$, $f(x) = y \equiv xfy \equiv (x, y) \in f$.
4. $f(x) = x^2 \equiv f = \{(x, x^2)\} \equiv f = \lambda x.x^2$.

1.2 Syntax

$$\begin{aligned} \mathit{Expr} ::= {} & \mathit{Num} \mid \\ & \mathit{Bool} \mid \\ & \triangle \mathit{Expr} \mid \\ & \mathit{Expr} \odot \mathit{Expr} \mid \\ & \texttt{if } \mathit{Expr} \texttt{ then } \mathit{Expr} \texttt{ else } \mathit{Expr} \end{aligned} \tag{1}$$

where Num is a predefined set of integer numbers (a.k.a. Z) and Bool is a predefined set of boolean values.

Convention: $e, e', \dots \in \mathit{Expr}$, $n, n' \in \mathit{Num}$, $b, b' \in \mathit{Bool}$, $v \in \mathit{Num} \cup \mathit{Bool}$ and $t \in \{\mathit{Number}, \mathit{Boolean}\}$.

1.3 Typing

$$\frac{}{\vdash n : \mathit{Number}} \tag{2}$$

$$\frac{}{\vdash b : \mathit{Boolean}} \tag{3}$$

$$\frac{\vdash e : \mathit{Number}}{\vdash \triangle e : \mathit{Number}} \tag{4}$$

$$\frac{\vdash e : \mathit{Number} \quad \vdash e' : \mathit{Number}}{\vdash e \odot e' : \mathit{Number}} \tag{5}$$

$$\frac{\vdash e : \mathit{Boolean} \quad \vdash e' : t \quad \vdash e'' : t}{\vdash \texttt{if } e \texttt{ then } e' \texttt{ else } e'' : t} \tag{6}$$

1.4 Big-Step Operational Semantics

$$\frac{}{n \mapsto n} \tag{7}$$

$$\frac{}{b \mapsto b} \tag{8}$$

$$\frac{e \mapsto n}{\triangle e \mapsto -n} \tag{9}$$

$$\frac{e \mapsto n \quad e' \mapsto n'}{e \odot e' \mapsto n + n'} \tag{10}$$

$$\frac{e \mapsto true \quad e' \mapsto v}{\texttt{if}\ e\ \texttt{then}\ e'\ \texttt{else}\ e'' \mapsto v} \tag{11}$$

$$\frac{e \mapsto false \quad e'' \mapsto v}{\texttt{if}\ e\ \texttt{then}\ e'\ \texttt{else}\ e'' \mapsto v} \tag{12}$$

1.5 Small-Step Operational Semantics

$$\frac{}{\triangle n \rightsquigarrow -n} \tag{13}$$

$$\frac{e \rightsquigarrow e'}{\triangle e \rightsquigarrow \triangle e'} \tag{14}$$

$$\frac{}{n \odot n' \rightsquigarrow n + n'} \tag{15}$$

$$\frac{e \rightsquigarrow e'}{e \odot e'' \rightsquigarrow e' \odot e''} \tag{16}$$

$$\frac{e' \rightsquigarrow e''}{e \odot e' \rightsquigarrow e \odot e''} \tag{17}$$

$$\frac{}{\texttt{if}\ true\ \texttt{then}\ e'\ \texttt{else}\ e'' \rightsquigarrow e'} \tag{18}$$

$$\frac{}{\texttt{if}\ false\ \texttt{then}\ e'\ \texttt{else}\ e'' \rightsquigarrow e''} \tag{19}$$

$$\frac{e \rightsquigarrow e'''}{\texttt{if}\ e\ \texttt{then}\ e'\ \texttt{else}\ e'' \rightsquigarrow \texttt{if}\ e'''\ \texttt{then}\ e'\ \texttt{else}\ e''} \tag{20}$$

$$\frac{e' \rightsquigarrow e'''}{\texttt{if}\ e\ \texttt{then}\ e'\ \texttt{else}\ e'' \rightsquigarrow \texttt{if}\ e\ \texttt{then}\ e'''\ \texttt{else}\ e''} \tag{21}$$

$$\frac{e'' \rightsquigarrow e'''}{\texttt{if}\ e\ \texttt{then}\ e'\ \texttt{else}\ e'' \rightsquigarrow \texttt{if}\ e\ \texttt{then}\ e'\ \texttt{else}\ e'''} \tag{22}$$

1.6 Properties

- Equivalence of the two semantics:

$$\forall e \in Expr\colon e \mapsto n \Leftrightarrow e \rightsquigarrow^* n. \tag{23}$$

- Type preservation:

$$\forall e, e' \in Expr\colon (\vdash e : t) \wedge e \rightsquigarrow e' \Rightarrow (\vdash e' : t). \tag{24}$$

- Progress:

$$\forall e \in Expr\colon (\vdash e : t) \Rightarrow e \in (Num \cup Bool) \vee \exists e' \in Expr\colon e \rightsquigarrow e'. \tag{25}$$

Type preservation + progress = soundness.

- Termination:

$$energy(v) = 0, \tag{26}$$
$$energy(\triangle e) = 1 + energy(e), \tag{27}$$
$$energy(e \odot e') = 1 + energy(e) + energy(e'), \tag{28}$$
$$energy(\texttt{if } e \texttt{ then } e' \texttt{ else } e'' = 1 + energy(e) + energy(e') \tag{29}$$
$$+ energy(e'''), \tag{30}$$

$$\forall e, e' \in Expr \colon e \rightsquigarrow e' \Rightarrow energy(e) > energy(e'), \tag{31}$$

$$\forall e \in Expr \colon energy(e) \in N \text{ and} \tag{32}$$

$$\forall e, e' \in Expr \colon e \rightsquigarrow e' \Rightarrow energy(e) > energy(e'). \tag{33}$$

- Determinism:

$$\forall v, v' \in (Num \cup Bool) \colon \forall e \in Expr \colon e \rightsquigarrow^* v \wedge e \rightsquigarrow^* v' \Rightarrow v = v'. \tag{34}$$

2 Seminar

1. Define a formal semantics of a simple imperative language with threads.
2. Given a grammar of the form $E = F_1, F_2, \ldots, F_n$, come up with an algorithmic way of generating the list of all non-isomorphic terms conforming to the grammar ordered by their depth.

3 Homework

$$\begin{aligned} Expr ::= {} & Num \mid \\ & Bool \mid \\ & \triangle Expr \mid \\ & Expr \texttt{ nand } Expr \mid \\ & \texttt{if } Expr \texttt{ then } Expr \texttt{ else } Expr \mid \\ & Var \\ Stmt ::= {} & Var \texttt{ = } Expr \\ Prg ::= {} & Stmt \mid \\ & Stmt\texttt{; } Prg \end{aligned} \tag{35}$$

$$\frac{}{\Gamma \vdash n : Number} \tag{36}$$

$$\frac{}{\Gamma \vdash b : Boolean} \tag{37}$$

$$\frac{\Gamma \vdash e : Number}{\Gamma \vdash \triangle e : Number} \tag{38}$$

$$\frac{\Gamma \vdash e : Boolean \quad \Gamma \vdash e' : Boolean}{\Gamma \vdash e \ \texttt{nand} \ e' : Boolean} \tag{39}$$

$$\frac{\Gamma \vdash e : Boolean \quad \Gamma \vdash e' : t \quad \Gamma \vdash e'' : t}{\Gamma \vdash \texttt{if} \ e \ \texttt{then} \ e' \ \texttt{else} \ e'' : t} \tag{40}$$

$$\frac{}{\Gamma \vdash v : \Gamma(v)} \tag{41}$$

$$\frac{\Gamma \vdash v : t \quad \Gamma \vdash e : t}{\Gamma \vdash v \ \texttt{=} \ e : \diamond} \tag{42}$$

$$\frac{v \notin dom(\Gamma) \quad \Gamma \vdash e : t}{\Gamma \vdash v \ \texttt{=} \ e : \diamond} \tag{43}$$

$$\frac{\Gamma \vdash v \ \texttt{=} \ e : \diamond \quad \Gamma \vdash e : t \quad \Gamma \cup \{(v : t)\} \vdash p : \diamond}{\Gamma \vdash v \ \texttt{=} \ e\texttt{;} \ p : \diamond} \tag{44}$$