Advanced Types

A4M36TPJ, 2013/2014

What is a type?

- What is a data type?
  - **1** is integer
  - **true** is boolean
  - Data type defines a set of possible values
- What is a type of an expression?
  - x + 5 : *int*, if x : *int* and + : *int x int -> int*

Sets vs. Types

- Q: What is the **difference** between the **set Bool** = {true, false} and the **Boolean type**?

- A: **Boolean** can be a **type of an expression**, e.g. **true and false**. This expression is not a member of the **set Bool**. On the other hand **true is a Boolean** and **false as well**. Hence, the **set of Boolean expressions is different from the set of Bool**.

Type Constructor

- We can construct **user-defined types** in many languages (Java, C++, …).

- To construct new types from existing types we use **type constructors**.

- For example **Int x Boolean** is a type constructor of a new **product type**. In Java, **class** is a constructor of a new class type.

First-order Type Systems

- Lack of type parametrization (Java generics)
- Lack type abstraction (Interfaces and abstract classes in Java)
- **Contains higher-order function types!**

Basic Setup

- $\Gamma \vdash \diamond$ ("Γ is a well-formed environment")
- $\Gamma \vdash A$ ("A is a well-formed type in environment Γ")
- $\Gamma \vdash e : A$ ("e is a well-formed term of type A in environment Γ")

$$\frac{}{\emptyset \vdash \diamond}$$

$$\frac{\Gamma \vdash A \quad a \notin dom(\Gamma)}{\Gamma \cup \{(a, A)\} \vdash \diamond}$$

$$\frac{\Gamma \vdash \diamond}{\Gamma \vdash x : \Gamma(x)}$$

$$\frac{\Gamma : \diamond \quad A \in Basic}{\Gamma \vdash A}$$

Function Type

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \to B}$$

$$\frac{\Gamma \cup \{(x, A)\} \vdash e : B}{\Gamma \vdash (\lambda x : A.e) : A \to B}$$

$$\frac{\Gamma \vdash e : A \to B \quad \Gamma \vdash p : A}{\Gamma \vdash e(p) : B}$$

Product Type

$$\frac{\Gamma \vdash A_1 \quad \Gamma \vdash A_2}{\Gamma \vdash A_1 \times A_2}$$

$$\frac{\Gamma \vdash e_1 : A_1 \quad \Gamma \vdash e_2 : A_2}{\Gamma \vdash (e_1, e_2) : A_1 \times A_2}$$

$$\frac{\Gamma \vdash e : A_1 \times A_2}{\Gamma \vdash \texttt{first}\ e : A_1}$$

$$\frac{\Gamma \vdash e : A_1 \times A_2}{\Gamma \vdash \texttt{second}\ e : A_2}$$

Tagged Union Type

$$\frac{\Gamma \vdash A_1 \quad \Gamma \vdash A_2}{\Gamma \vdash A_1 + A_2}$$

$$\frac{\Gamma \vdash e : A_1 \quad \Gamma \vdash A_2}{\Gamma \vdash \texttt{inLeft}_{A_2}\, e : A_1 + A_2}$$

$$\frac{\Gamma \vdash A_1 \quad \Gamma \vdash e : A_2}{\Gamma \vdash \texttt{inRight}_{A_1}\, e : A_1 + A_2}$$

Tagged Union Type

$$\frac{\Gamma \vdash e : A_1 + A_2}{\Gamma \vdash \texttt{isLeft}\ \ e : Boolean}$$

$$\frac{\Gamma \vdash e : A_1 + A_2}{\Gamma \vdash \texttt{asLeft}\ \ e : A_1}$$

$$\frac{\Gamma \vdash e : A_1 + A_2}{\Gamma \vdash \texttt{isRight}\ \ e : Boolean}$$

$$\frac{\Gamma \vdash e : A_1 + A_2}{\Gamma \vdash \texttt{asRight}\ \ e : A_2}$$

Record Type

$$\frac{\Gamma \vdash A_1 \quad \ldots \quad \Gamma \vdash A_n}{\Gamma \vdash \texttt{Record}(l_1 : A_1, \ldots, l_n : A_n)}$$

$$\frac{\Gamma \vdash e_1 : A_1 \quad \ldots \quad \Gamma \vdash e_n : A_n}{\Gamma \vdash \texttt{record}(l_1\texttt{=}e_1, \ldots, l_n\texttt{=}e_n) : \texttt{Record}(l_1 : A_1, \ldots, l_n : A_n)}$$

$$\frac{\Gamma \vdash e : \texttt{Record}(l_1 : A_1, \ldots, l_j : A_j, \ldots, l_n : A_n)}{e.l_j : A_j}$$

Reference Type

$$\frac{\Gamma \vdash A}{\Gamma \vdash \texttt{Ref}\ \ A}$$

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash \texttt{ref}\ \ e : \texttt{Ref}\ \ A}$$

$$\frac{\Gamma \vdash e : \texttt{Ref}\ \ A}{\Gamma \vdash \texttt{deref}\ \ e : A}$$

$$\frac{\Gamma \vdash e : \texttt{Ref}\ \ A \quad \Gamma \vdash e' : A}{\Gamma \vdash e\ \texttt{=}\ e' : \diamond}$$

Type Variables

$$\frac{\Gamma \vdash \diamond \quad X \notin dom(\Gamma)}{\Gamma \cup \{X\} \vdash \diamond}$$

Recursive Type

$$\frac{\Gamma \cup \{X\} \vdash A}{\Gamma \vdash \mu X.A}$$

$$\frac{\Gamma \vdash e : \mu X.A}{\Gamma \vdash \texttt{unfold e} : A[X \mapsto \mu X.A]}$$

$$\frac{\Gamma \vdash e : A[X \mapsto \mu X.A]}{\Gamma \vdash \texttt{fold e} : \mu X.A}$$

Example Language F_1

Syntax

A,B ::=		types
	K	$K \in Basic$ basic types
	$A \to B$	function types
M,N ::=		terms
	x	variable
	$\lambda x{:}A.M$	function
	$M\ N$	application

Example Language F_1

Type System

(Env $\emptyset$)

$$\frac{}{\emptyset \vdash \diamond}$$

(Env x)

$$\frac{\Gamma \vdash A \qquad x \notin dom(\Gamma)}{\Gamma, x{:}A \vdash \diamond}$$

(Type Const)

$$\frac{\Gamma \vdash \diamond \qquad K \in Basic}{\Gamma \vdash K}$$

(Type Arrow)

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \to B}$$

(Val x)

$$\frac{\Gamma', x{:}A, \Gamma'' \vdash \diamond}{\Gamma', x{:}A, \Gamma'' \vdash x{:}A}$$

(Val Fun)

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : A \to B}$$

(Val Appl)

$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B}$$

F_1 - Basic Types

(Type Unit)

$$\frac{\Gamma \vdash \diamond}{\Gamma \vdash \mathit{Unit}}$$

(Val Unit)

$$\frac{\Gamma \vdash \diamond}{\Gamma \vdash \mathit{unit} : \mathit{Unit}}$$

(Type Bool)

$$\frac{\Gamma \vdash \diamond}{\Gamma \vdash \mathit{Bool}}$$

(Val True)

$$\frac{\Gamma \vdash \diamond}{\Gamma \vdash \mathit{true} : \mathit{Bool}}$$

(Val False)

$$\frac{\Gamma \vdash \diamond}{\Gamma \vdash \mathit{false} : \mathit{Bool}}$$

(Val Cond)

$$\frac{\Gamma \vdash M : \mathit{Bool} \qquad \Gamma \vdash N_1 : A \qquad \Gamma \vdash N_2 : A}{\Gamma \vdash (\mathit{if}_A\ M\ \mathit{then}\ N_1\ \mathit{else}\ N_2) : A}$$

F_1 - Basic Types

(Type Nat)
$$\frac{\Gamma \vdash \diamond}{\Gamma \vdash Nat}$$

(Val Zero)
$$\frac{\Gamma \vdash \diamond}{\Gamma \vdash 0 : Nat}$$

(Val Succ)
$$\frac{\Gamma \vdash M : Nat}{\Gamma \vdash succ\ M : Nat}$$

(Val Pred)
$$\frac{\Gamma \vdash M : Nat}{\Gamma \vdash pred\ M : Nat}$$

(Val IsZero)
$$\frac{\Gamma \vdash M : Nat}{\Gamma \vdash isZero\ M : Bool}$$

Universal Type

$$\frac{\Gamma \cup \{X\} \vdash A}{\Gamma \vdash \forall X.A}$$

$$\frac{\Gamma \cup \{X\} \vdash e : A}{\Gamma \vdash \lambda X.e : \forall X.A}$$

$$\frac{\Gamma \vdash e : \forall X.A \quad \Gamma \vdash B}{\Gamma \vdash e(B) : A[X \mapsto B]}$$

Subtype Polymorphism

- We define a new binary relation <: on types and a new judgement: Γ ⊢ A <: B (“A is a subtype of B in environment Γ”)

$$\frac{}{\Gamma \vdash A <: A}$$

$$\frac{\Gamma \vdash A <: B \quad \Gamma \vdash B <: C}{\Gamma \vdash A <: C}$$

Subsumption

$$\frac{\Gamma \vdash e : A \quad \Gamma \vdash A <: B}{\Gamma \vdash e : B}$$

Top Type

$$\frac{}{\Gamma \vdash Top} \quad \Gamma \vdash \diamond$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A <: Top}$$

Subtyping

$$\frac{\Gamma \vdash A' <: A \quad \Gamma \vdash B <: B'}{\Gamma \vdash A \to B <: A' \to B'}$$

$$\frac{\Gamma \vdash A' <: A \quad \Gamma \vdash B' <: B}{\Gamma \vdash A' \times B' <: A \times B}$$

$$\frac{\Gamma \vdash A' <: A \quad \Gamma \vdash B' <: B}{\Gamma \vdash A' + B' <: A + B}$$

$$\frac{\Gamma \vdash A'_1 <: A_1 \quad \ldots \quad \Gamma \vdash A'_n <: A_n \quad \Gamma \vdash A'_{n+1} \quad \ldots \quad \Gamma \vdash A'_{n+m}}{\Gamma \vdash \texttt{Record}(l_1 : A'_1, \ldots, l_{n+m} : A'_{n+m}) <: \texttt{Record}(l_1 : A_1, \ldots, l_n : A_n)}$$

Bounded Type Variables

$$\frac{\Gamma \vdash A \quad X \notin dom(\Gamma)}{\Gamma \cup \{X <: A\} \vdash \diamond}$$

$$\frac{\Gamma \cup \{X <: A\} \vdash \diamond}{\Gamma \cup \{X <: A\} \vdash X}$$

$$\frac{\Gamma \cup \{X <: A\} \vdash \diamond}{\Gamma \cup \{X <: A\} \vdash X <: A}$$

Subtyping of Recursive Types

$$\frac{\Gamma \cup \{X <: Top\} \vdash A}{\Gamma \vdash \mu X.A}$$

$$\frac{\Gamma \vdash \mu X.A \quad \Gamma \vdash \mu Y.B \quad \Gamma \cup \{Y <: Top, X <: Y\} \vdash A <: B}{\Gamma \vdash \mu X.A <: \mu Y.B}$$

Subtyping of Universal Types

$$\frac{\Gamma \cup \{X <: A\} \vdash B}{\Gamma \vdash \forall X <: A.B}$$

$$\frac{\Gamma \vdash A' <: A \quad \Gamma \cup \{X <: A'\} \vdash B <: B'}{\Gamma \vdash (\forall X <: A.B) <: (\forall X <: A'.B')}$$

$$\frac{\Gamma \cup \{X <: A\} \vdash e : B}{\Gamma \vdash \lambda X <: A.e : \forall X <: A.B}$$

$$\frac{\Gamma \vdash e : \forall X <: A.B \quad \Gamma \vdash A' <: A}{\Gamma \vdash e(A') : B[X \mapsto A']}$$

Example Language $F_{2<:}$

Syntax

A,B	::=	types
	X	type variable
	Top	the biggest type
	$A \to B$	function type
	$\forall X<:A.B$	bounded universally quantified type
M,N	::=	terms
	x	variable
	$\lambda x:A.M$	function
	$M\ N$	application
	$\lambda X<:A.M$	bounded polymorphic abstraction
	$M\ A$	type instantiation

(Env $\emptyset$)

$$\frac{}{\emptyset \vdash \diamond}$$

(Env x)

$$\frac{\Gamma \vdash A \qquad x \notin dom(\Gamma)}{\Gamma, x{:}A \vdash \diamond}$$

(Env X)

$$\frac{\Gamma \vdash \diamond \qquad X \notin dom(\Gamma)}{\Gamma, X \vdash \diamond}$$

(Type X)

$$\frac{\Gamma', X, \Gamma'' \vdash \diamond}{\Gamma', X, \Gamma'' \vdash X}$$

(Type Arrow)

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \to B}$$

(Type Forall)

$$\frac{\Gamma, X \vdash A}{\Gamma \vdash \forall X.A}$$

(Val x)

$$\frac{\Gamma', x{:}A, \Gamma'' \vdash \diamond}{\Gamma', x{:}A, \Gamma'' \vdash x{:}A}$$

(Val Fun)

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : A \to B}$$

(Val Appl)

$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B}$$

(Val Fun2)

$$\frac{\Gamma, X \vdash M : A}{\Gamma \vdash \lambda X.M : \forall X.A}$$

(Val Appl2)

$$\frac{\Gamma \vdash M : \forall X.A \qquad \Gamma \vdash B}{\Gamma \vdash M\ B : [B/X]A}$$

(Type Forall<:)

$$\frac{\Gamma, X<:A \vdash B}{\Gamma \vdash \forall X<:A.B}$$

(Sub Forall<:)

$$\frac{\Gamma \vdash A' <: A \qquad \Gamma, X<:A' \vdash B <: B'}{\Gamma \vdash (\forall X<:A.B) <: (\forall X<:A'.B')}$$

(Val Fun2<:)

$$\frac{\Gamma, X<:A \vdash M : B}{\Gamma \vdash \lambda X<:A.M : \forall X<:A.B}$$

(Val Appl2<:)

$$\frac{\Gamma \vdash M : \forall X<:A.B \qquad \Gamma \vdash A' <: A}{\Gamma \vdash M\ A' : [A'/X]B}$$