

Advanced Types

A4M36TPJ, 2015/2016

What is a type?

- What is a data type?
 - **1** is the integer
 - **true** is the boolean
 - Data type **defines a set of possible values**
- What is a type of an expression?
 - $x + 5 : int$, if $x : int$ and $+ : int \times int \rightarrow int$

Sets vs. Types

- Q: What is the **difference** between the **set Bool** = {true, false} and the **Boolean type**?
- A: **Boolean** can be a **type of an expression**, e.g. **true and false**. This expression is not a member of the **set Bool**. On the other hand **true is a Boolean** and **false as well**. Hence, the **set of Boolean expressions is different from the set of Bool**.

Type Constructor

- We can construct **user-defined types** in many languages (Java, C++, ...).
- To construct new types from existing types we use **type constructors**.
- For example **Int x Boolean** is a type constructor of a new **product type**. In Java, **class** is a constructor of a new class type.

First-order Type Systems

- Lack of type parametrization (Java generics)
- Lack type abstraction (Interfaces and abstract classes in Java)
- **Contains higher-order function types!**

Basic Setup

- $\Gamma \vdash \diamond$ (“ Γ is a well-formed environment”)
- $\Gamma \vdash A$ (“ A is a well-formed type in environment Γ ”)
- $\Gamma \vdash e : A$ (“ e is a well-formed term of type A in environment Γ ”)

$$\overline{\emptyset \vdash \diamond}$$

$$\frac{\Gamma \vdash A \quad a \notin \text{dom}(\Gamma)}{\Gamma \cup \{(a, A)\} \vdash \diamond}$$

$$\frac{\Gamma \vdash \diamond}{\Gamma \vdash x : \Gamma(x)}$$

$$\frac{\Gamma : \diamond \quad A \in \text{Basic}}{\Gamma \vdash A}$$

Function Type

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \rightarrow B}$$

$$\frac{\Gamma \cup \{(x, A)\} \vdash e : B}{\Gamma \vdash (\lambda x : A. e) : A \rightarrow B}$$

$$\frac{\Gamma \vdash e : A \rightarrow B \quad \Gamma \vdash p : A}{\Gamma \vdash e(p) : B}$$

Product Type

$$\frac{\Gamma \vdash A_1 \quad \Gamma \vdash A_2}{\Gamma \vdash A_1 \times A_2}$$

$$\frac{\Gamma \vdash e_1 : A_1 \quad \Gamma \vdash e_2 : A_2}{\Gamma \vdash (e_1, e_2) : A_1 \times A_2}$$

$$\frac{\Gamma \vdash e : A_1 \times A_2}{\Gamma \vdash \mathbf{first} \ e : A_1}$$

$$\frac{\Gamma \vdash e : A_1 \times A_2}{\Gamma \vdash \mathbf{second} \ e : A_2}$$

Tagged Union Type

$$\frac{\Gamma \vdash A_1 \quad \Gamma \vdash A_2}{\Gamma \vdash A_1 + A_2}$$

$$\frac{\Gamma \vdash e : A_1 \quad \Gamma \vdash A_2}{\Gamma \vdash \text{inLeft}_{A_2} e : A_1 + A_2}$$

$$\frac{\Gamma \vdash A_1 \quad \Gamma \vdash e : A_2}{\Gamma \vdash \text{inRight}_{A_1} e : A_1 + A_2}$$

Tagged Union Type

$$\frac{\Gamma \vdash e : A_1 + A_2}{\Gamma \vdash \text{isLeft } e : \textit{Boolean}}$$

$$\frac{\Gamma \vdash e : A_1 + A_2}{\Gamma \vdash \text{isRight } e : \textit{Boolean}}$$

$$\frac{\Gamma \vdash e : A_1 + A_2}{\Gamma \vdash \text{asLeft } e : A_1}$$

$$\frac{\Gamma \vdash e : A_1 + A_2}{\Gamma \vdash \text{asRight } e : A_2}$$

Record Type

$$\frac{\Gamma \vdash A_1 \quad \dots \quad \Gamma \vdash A_n}{\Gamma \vdash \text{Record}(l_1 : A_1, \dots, l_n : A_n)}$$

$$\frac{\Gamma \vdash e_1 : A_1 \quad \dots \quad \Gamma \vdash e_n : A_n}{\Gamma \vdash \text{record}(l_1=e_1, \dots, l_n=e_n) : \text{Record}(l_1 : A_1, \dots, l_n : A_n)}$$

$$\frac{\Gamma \vdash e : \text{Record}(l_1 : A_1, \dots, l_j : A_j, \dots, l_n : A_n)}{e.l_j : A_j}$$

Reference Type

$$\frac{\Gamma \vdash A}{\Gamma \vdash \text{Ref } A}$$

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash \text{ref } e : \text{Ref } A}$$

$$\frac{\Gamma \vdash e : \text{Ref } A}{\Gamma \vdash \text{deref } e : A}$$

$$\frac{\Gamma \vdash e : \text{Ref } A \quad \Gamma \vdash e' : A}{\Gamma \vdash e = e' : \diamond}$$

Type Variables

$$\frac{\Gamma \vdash \diamond \quad X \notin \text{dom}(\Gamma)}{\Gamma \cup \{X\} \vdash \diamond}$$

Recursive Type

$$\frac{\Gamma \cup \{X\} \vdash A}{\Gamma \vdash \mu X.A}$$

$$\frac{\Gamma \vdash e : \mu X.A}{\Gamma \vdash \text{unfold } e : A[X \mapsto \mu X.A]}$$

$$\frac{\Gamma \vdash e : A[X \mapsto \mu X.A]}{\Gamma \vdash \text{fold } e : \mu X.A}$$

Example Language F_1

Syntax

$A, B ::=$		types
K	$K \in Basic$	basic types
$A \rightarrow B$		function types
$M, N ::=$		terms
x		variable
$\lambda x:A.M$		function
$M N$		application

Example Language F_1

Type System

(Env \emptyset)

$$\frac{}{\emptyset \vdash \diamond}$$

(Env x)

$$\frac{\Gamma \vdash A \quad x \notin \text{dom}(\Gamma)}{\Gamma, x:A \vdash \diamond}$$

(Type Const)

$$\frac{\Gamma \vdash \diamond \quad K \in \text{Basic}}{\Gamma \vdash K}$$

(Type Arrow)

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \rightarrow B}$$

(Val x)

$$\frac{\Gamma', x:A, \Gamma'' \vdash \diamond}{\Gamma', x:A, \Gamma'' \vdash x:A}$$

(Val Fun)

$$\frac{\Gamma, x:A \vdash M : B}{\Gamma \vdash \lambda x:A.M : A \rightarrow B}$$

(Val Appl)

$$\frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash M N : B}$$

F₁ - Basic Types

(Type Unit)

$$\Gamma \vdash \diamond$$

$$\Gamma \vdash \mathit{Unit}$$

(Val Unit)

$$\Gamma \vdash \diamond$$

$$\Gamma \vdash \mathit{unit} : \mathit{Unit}$$

(Type Bool)

$$\Gamma \vdash \diamond$$

$$\Gamma \vdash \mathit{Bool}$$

(Val True)

$$\Gamma \vdash \diamond$$

$$\Gamma \vdash \mathit{true} : \mathit{Bool}$$

(Val False)

$$\Gamma \vdash \diamond$$

$$\Gamma \vdash \mathit{false} : \mathit{Bool}$$

(Val Cond)

$$\Gamma \vdash M : \mathit{Bool} \quad \Gamma \vdash N_1 : A \quad \Gamma \vdash N_2 : A$$

$$\Gamma \vdash (\mathit{if}_A M \mathit{then} N_1 \mathit{else} N_2) : A$$

F₁ - Basic Types

(Type Nat)

$$\Gamma \vdash \diamond$$

$$\Gamma \vdash \mathit{Nat}$$

(Val Zero)

$$\Gamma \vdash \diamond$$

$$\Gamma \vdash 0 : \mathit{Nat}$$

(Val Succ)

$$\Gamma \vdash M : \mathit{Nat}$$

$$\Gamma \vdash \mathit{succ } M : \mathit{Nat}$$

(Val Pred)

$$\Gamma \vdash M : \mathit{Nat}$$

$$\Gamma \vdash \mathit{pred } M : \mathit{Nat}$$

(Val IsZero)

$$\Gamma \vdash M : \mathit{Nat}$$

$$\Gamma \vdash \mathit{isZero } M : \mathit{Bool}$$

Universal Type

$$\frac{\Gamma \cup \{X\} \vdash A}{\Gamma \vdash \forall X.A}$$

$$\frac{\Gamma \cup \{X\} \vdash e : A}{\Gamma \vdash \lambda X.e : \forall X.A}$$

$$\frac{\Gamma \vdash e : \forall X.A \quad \Gamma \vdash B}{\Gamma \vdash e(B) : A[X \mapsto B]}$$

Subtype Polymorphism

- We define a new binary relation $<:$ on types and a new judgement: $\Gamma \vdash A <: B$ (“A is a subtype of B in environment Γ ”)

$$\frac{}{\Gamma \vdash A <: A}$$

$$\frac{\Gamma \vdash A <: B \quad \Gamma \vdash B <: C}{\Gamma \vdash A <: C}$$

Subsumption

$$\frac{\Gamma \vdash e : A \quad \Gamma \vdash A <: B}{\Gamma \vdash e : B}$$

Top Type

$$\frac{\Gamma \vdash \diamond}{\Gamma \vdash \mathit{Top}}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A <: \mathit{Top}}$$

Subtyping

$$\frac{\Gamma \vdash A' <: A \quad \Gamma \vdash B <: B'}{\Gamma \vdash A \rightarrow B <: A' \rightarrow B'}$$

$$\frac{\Gamma \vdash A' <: A \quad \Gamma \vdash B' <: B}{\Gamma \vdash A' \times B' <: A \times B}$$

$$\frac{\Gamma \vdash A' <: A \quad \Gamma \vdash B' <: B}{\Gamma \vdash A' + B' <: A + B}$$

$$\frac{\Gamma \vdash A'_1 <: A_1 \quad \dots \quad \Gamma \vdash A'_n <: A_n \quad \Gamma \vdash A'_{n+1} \quad \dots \quad \Gamma \vdash A'_{n+m}}{\Gamma \vdash \text{Record}(l_1 : A'_1, \dots, l_{n+m} : A'_{n+m}) <: \text{Record}(l_1 : A_1, \dots, l_n : A_n)}$$

Bounded Type Variables

$$\frac{\Gamma \vdash A \quad X \notin \text{dom}(\Gamma)}{\Gamma \cup \{X <: A\} \vdash \diamond}$$

$$\frac{\Gamma \cup \{X <: A\} \vdash \diamond}{\Gamma \cup \{X <: A\} \vdash X}$$

$$\frac{\Gamma \cup \{X <: A\} \vdash \diamond}{\Gamma \cup \{X <: A\} \vdash X <: A}$$

Subtyping of Recursive Types

$$\frac{\Gamma \cup \{X <: Top\} \vdash A}{\Gamma \vdash \mu X.A}$$

$$\frac{\Gamma \vdash \mu X.A \quad \Gamma \vdash \mu Y.B \quad \Gamma \cup \{Y <: Top, X <: Y\} \vdash A <: B}{\Gamma \vdash \mu X.A <: \mu Y.B}$$

Subtyping of Universal Types

$$\frac{\Gamma \cup \{X <: A\} \vdash B}{\Gamma \vdash \forall X <: A. B}$$

$$\frac{\Gamma \vdash A' <: A \quad \Gamma \cup \{X <: A'\} \vdash B <: B'}{\Gamma \vdash (\forall X <: A. B) <: (\forall X <: A'. B')}$$

$$\frac{\Gamma \cup \{X <: A\} \vdash e : B}{\Gamma \vdash \lambda X <: A. e : \forall X <: A. B}$$

$$\frac{\Gamma \vdash e : \forall X <: A. B \quad \Gamma \vdash A' <: A}{\Gamma \vdash e(A') : B[X \mapsto A']}$$

Example Language $F_{2<}$:

Syntax

$A, B ::=$

X

Top

$A \rightarrow B$

$\forall X <: A. B$

types

type variable

the biggest type

function type

bounded universally quantified type

$M, N ::=$

x

$\lambda x : A. M$

$M N$

$\lambda X <: A. M$

$M A$

terms

variable

function

application

bounded polymorphic abstraction

type instantiation

$$\begin{array}{c}
\text{(Env } \emptyset) \\
\hline
\emptyset \vdash \diamond
\end{array}
\quad
\begin{array}{c}
\text{(Env } x) \\
\frac{\Gamma \vdash A \quad x \notin \text{dom}(\Gamma)}{\Gamma, x:A \vdash \diamond}
\end{array}
\quad
\begin{array}{c}
\text{(Env } X) \\
\hline
\Gamma \vdash \diamond \quad X \notin \text{dom}(\Gamma) \\
\hline
\Gamma, X \vdash \diamond
\end{array}$$

$$\begin{array}{c}
\text{(Type } X) \\
\frac{\Gamma', X, \Gamma'' \vdash \diamond}{\Gamma', X, \Gamma'' \vdash X}
\end{array}
\quad
\begin{array}{c}
\text{(Type Arrow)} \\
\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \rightarrow B}
\end{array}
\quad
\begin{array}{c}
\text{(Type Forall)} \\
\frac{\Gamma, X \vdash A}{\Gamma \vdash \forall X.A}
\end{array}$$

$$\begin{array}{c}
\text{(Val } x) \\
\frac{\Gamma', x:A, \Gamma'' \vdash \diamond}{\Gamma', x:A, \Gamma'' \vdash x:A}
\end{array}
\quad
\begin{array}{c}
\text{(Val Fun)} \\
\frac{\Gamma, x:A \vdash M : B}{\Gamma \vdash \lambda x:A.M : A \rightarrow B}
\end{array}
\quad
\begin{array}{c}
\text{(Val Appl)} \\
\frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash M N : B}
\end{array}$$

$$\begin{array}{c}
\text{(Val Fun2)} \\
\frac{\Gamma, X \vdash M : A}{\Gamma \vdash \lambda X.M : \forall X.A}
\end{array}
\quad
\begin{array}{c}
\text{(Val Appl2)} \\
\frac{\Gamma \vdash M : \forall X.A \quad \Gamma \vdash B}{\Gamma \vdash M B : [B/X]A}
\end{array}$$

(Type Forall<:)

$$\Gamma, X <: A \vdash B$$

$$\Gamma \vdash \forall X <: A. B$$

(Sub Forall<:)

$$\Gamma \vdash A' <: A \quad \Gamma, X <: A' \vdash B <: B'$$

$$\Gamma \vdash (\forall X <: A. B) <: (\forall X <: A'. B')$$

(Val Fun2<:)

$$\Gamma, X <: A \vdash M : B$$

$$\Gamma \vdash \lambda X <: A. M : \forall X <: A. B$$

(Val Appl2<:)

$$\Gamma \vdash M : \forall X <: A. B \quad \Gamma \vdash A' <: A$$

$$\Gamma \vdash M A' : [A'/X]B$$