Denotational Semantics

A4M36TPJ, 2015/2016

Operational Semantics

- Usually **well suited for reasoning about whole programs**, less than ideal for reasoning about program fragments.
- Sometimes tends to **overspecify the implementation** of certain language features (e.g. evaluation order).
- Tends to put **emphasis on syntax** (rather than semantics) of the language.

Denotational Semantics

- Meaning of a program can be determined from the **meaning of its parts**.
- Unlike an operational semantics, a denotational semantics emphasizes **what** the meaning of a phrase is, not **how** the phrase is evaluated.

Denotational Semantics

- Consists of three parts:
  - **Syntactic Algebra**
  - **Semantic Algebra**
  - **Meaning Function**

Syntactic Algebra

- Describes **abstract syntax of the language**.
- Can be **specified by a grammar**.

Semantic Algebra

- Models the **meaning of program phrases**.
- Consists of a **collection of semantic domains** along with functions that manipulate these domains.
- The meaning of a program may be as simple as an element of a primitive semantic domain like Int, the domain of integers.
- More typically, the meaning of a program is an element of a **function domain** that maps **context domains** to an **answer domain**.

Context Domains

- **Denotational analogue** of state components.
- Model such entities as **name/value associations**, **contents of memory**, and **control information**.

Answer Domains

- Represent the **possible meanings of programs**.
- Include components that model **context information** that was transformed.

Meaning Function

- Maps elements of the **syntactic algebra** (i.e., nodes in the abstract syntax trees) to their meanings in the **semantic algebra**.
- Usually a collection of so-called **valuation** functions, one for each **syntactic domain** defined by the **abstract syntax** for the language.
- The function must be a **homomorphism** between the syntactic algebra and the semantic algebra.

Meaning Function

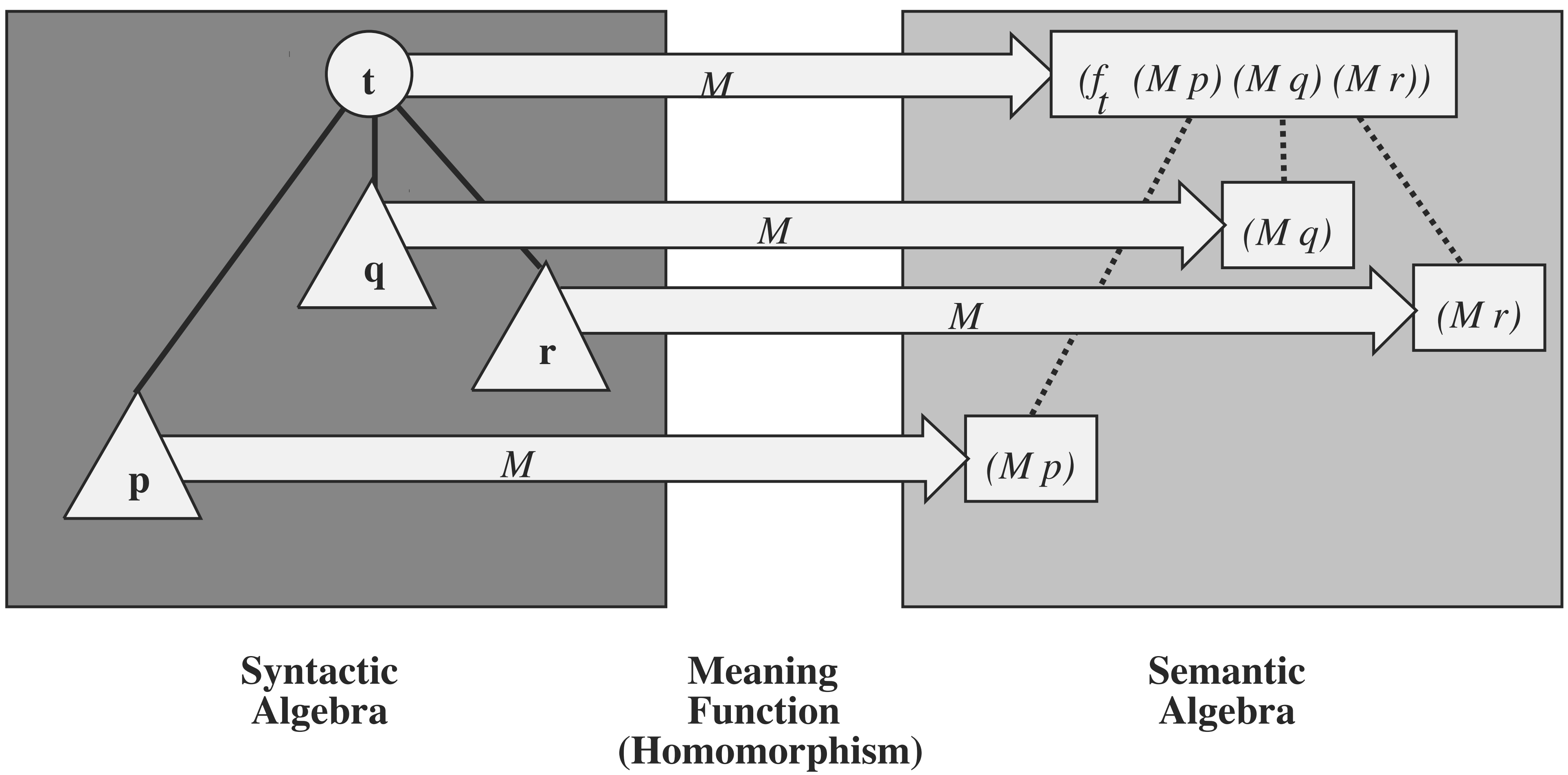
Suppose **M** is a meaning function and **t** is a node in an **abstract syntax tree**, with children **t_1, . . . , t_k**. Then

$$(M\ t) = (f_t\ (M\ t_1)\ ...\ (M\ t_k))$$

where **f_t** is a function that is **determined** by the **syntactic class of t**.

The reason to restrict meaning functions to homomorphisms is that their **structure-preserving behavior greatly simplifies reasoning**.

Denotational Semantics Game



Expression Language

Syntax

$$
\begin{aligned}
Expr ::= \; & Num \mid \\
& \triangle Expr \mid \\
& Expr \odot Expr
\end{aligned}
$$

Semantics

Semantic domain: N.

$$[\![n]\!] = n$$

$$[\![\triangle e]\!] = [\![\triangle]\!]([\![e]\!])$$

$$[\![\triangle]\!] = \lambda x. - x \text{ (i.e. unary minus)}$$

$$[\![e_1 \odot e_2]\!] = [\![\odot]\!]([\![e_1]\!], [\![e_2]\!])$$

$$[\![\odot]\!] = \lambda x, y.x + y \text{ (i.e. plus)}$$

Logic Formulae Language

Syntax

$$
\begin{aligned}
Formula ::= {} & true \mid \\
& false \mid \\
& \neg Formula \mid \\
& Formula\ BinaryConnective\ Formula \\
BinaryConnective ::= {} & \wedge \mid \vee
\end{aligned}
$$

Semantics

Semantic domain: $\{0, 1\}$.

$$[\![true]\!] = 1$$

$$[\![false]\!] = 0$$

$$[\![\neg f]\!] = [\![\neg]\!]([\![f]\!])$$

$$[\![\neg]\!] = \lambda x.1 - x$$

Semantics

$$[\![f_1 \ c \ f_2]\!] = [\![c]\!]([\![f_1]\!], [\![f_2]\!]) \quad (c \in BinaryConnective)$$

$$[\![\wedge]\!] = \lambda x, y.x \cdot y$$

$$[\![\vee]\!] = \lambda x, y.(x + y) - (x \cdot y)$$

Regular Expressions

Syntax

$$
\begin{aligned}
RegExp ::= {} & \emptyset \mid \\
& \epsilon \mid \\
& A \mid \\
& RegExp^* \mid \\
& RegExp\ BinOp\ RegExp \\
BinOp ::= {} & + \mid \cdot
\end{aligned}
$$

where A is a predefined set of characters (alphabet).

Semantics

Semantic domain: A^*.

$$[\![\emptyset]\!] = \{\}$$

$$[\![\epsilon]\!] = \{\epsilon\}$$

$$[\![a]\!] = \{a\}$$

$$[\![e^*]\!] = [\![^*]\!]([\![e]\!])$$

Semantics

$$[\![*]\!] = \lambda L.\{l_1 \cdot \ldots \cdot l_n | n \in N \wedge l_i \in L\} \quad \text{(note: including } \epsilon\text{)}$$

$$[\![e_1 \; o \; e_2]\!] = [\![o]\!]([\![e_1]\!], [\![e_2]\!]) \quad (o \in BinOp)$$

$$[\![+]\!] = \cup$$

$$[\![\cdot]\!] = \lambda A, B.\{a \cdot b | a \in A \wedge b \in B\}$$

Lambda Calculus

Syntax

$$
\begin{aligned}
Expr ::= \; & X \mid \\
& \lambda X.Expr \mid \\
& Expr \; Expr
\end{aligned}
$$

Semantics

Semantic domains: $env = string \rightarrow function$, $fcn = fcn \rightarrow fcn$; notational conventions $e \in env, f, f' \in fcn, E, E' \in Expr$

$$[\![x]\!] = \lambda e.e(x) \tag{26}$$

$$[\![\lambda x.E]\!] = \lambda e.\lambda p.[\![E]\!](e[x \mapsto p]) \tag{27}$$

$$[\![E_1\ E_2]\!] = \lambda e.([\![E_1]\!](e))([\![E_2]\!](e)) \tag{28}$$