Semantics Properties

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Global Properties of a Programming Language

- **universality**: the language can express all computable programs;
- **determinism**: the set of possible outcomes from executing a program on any particular inputs is a singleton;
- **strong normalization**: all programs are guaranteed to terminate on all inputs (i.e., it is not possible to express an infinite loop);
- **static checkability**: a class of program errors can be found by static analysis without resorting to execution;
- **referential transparency**: different occurrences of an expression within the same context always have the same meaning.

SOS Formal Definition

$$S =< CF, \Rightarrow, FC, IF, OF >$$

- CF is a set of configurations,
- $\Rightarrow$ is a binary relation over configurations ($\Rightarrow \subseteq CF \times CF$),
- FC is a set of final configurations ($FC \subseteq CF$),
- $IF : Program \times Input \rightarrow CF$ is an input function and
- $OF : FC \rightarrow Output$ is an output function.

Syntax

$$
\begin{aligned}
Expr ::= \; & Num \mid \\
& Bool \mid \\
& \triangle Expr \mid \\
& Expr \odot Expr \mid \\
& \texttt{if}\ Expr\ \texttt{then}\ Expr\ \texttt{else}\ Expr
\end{aligned}
$$

where Num is a predefined set of integer numbers (a.k.a. Z) and $Bool$ is a predefined set of boolean values.

Convention: $e, e', \cdots \in Expr$, $n, n' \in Num$, $b, b' \in Bool$, $v \in Num \cup Bool$ and $t \in \{Number, Boolean\}$.

Type System

$$\frac{}{\vdash n : Number}$$

$$\frac{}{\vdash b : Boolean}$$

$$\frac{\vdash e : Number}{\vdash \triangle e : Number}$$

$$\frac{\vdash e : Number \quad \vdash e' : Number}{\vdash e \odot e' : Number}$$

$$\frac{\vdash e : Boolean \quad \vdash e' : t \quad \vdash e'' : t}{\vdash \texttt{if}\ e\ \texttt{then}\ e'\ \texttt{else}\ e'' : t}$$

BOS

$$\frac{}{n \mapsto n}$$

$$\frac{}{b \mapsto b}$$

$$\frac{e \mapsto n}{\triangle e \mapsto -n}$$

$$\frac{e \mapsto n \quad e' \mapsto n'}{e \odot e' \mapsto n + n'}$$

$$\frac{e \mapsto true \quad e' \mapsto v}{\texttt{if}\ e\ \texttt{then}\ e'\ \texttt{else}\ e'' \mapsto v}$$

$$\frac{e \mapsto false \quad e'' \mapsto v}{\texttt{if}\ e\ \texttt{then}\ e'\ \texttt{else}\ e'' \mapsto v}$$

SOS

$$\frac{}{\triangle n \rightsquigarrow -n}$$

$$\frac{e \rightsquigarrow e'}{\triangle e \rightsquigarrow \triangle e'}$$

$$\frac{}{n \odot n' \rightsquigarrow n + n'}$$

$$\frac{e \rightsquigarrow e'}{e \odot e'' \rightsquigarrow e' \odot e''}$$

$$\frac{e' \rightsquigarrow e''}{e \odot e' \rightsquigarrow e \odot e''}$$

SOS

$$\frac{}{\texttt{if}\ \mathit{true}\ \texttt{then}\ e'\ \texttt{else}\ e'' \rightsquigarrow e'}$$

$$\frac{}{\texttt{if}\ \mathit{false}\ \texttt{then}\ e'\ \texttt{else}\ e'' \rightsquigarrow e''}$$

$$\frac{e \rightsquigarrow e'''}{\texttt{if}\ e\ \texttt{then}\ e'\ \texttt{else}\ e'' \rightsquigarrow \texttt{if}\ e'''\ \texttt{then}\ e'\ \texttt{else}\ e''}$$

$$\frac{e' \rightsquigarrow e'''}{\texttt{if}\ e\ \texttt{then}\ e'\ \texttt{else}\ e'' \rightsquigarrow \texttt{if}\ e\ \texttt{then}\ e'''\ \texttt{else}\ e''}$$

$$\frac{e'' \rightsquigarrow e'''}{\texttt{if}\ e\ \texttt{then}\ e'\ \texttt{else}\ e'' \rightsquigarrow \texttt{if}\ e\ \texttt{then}\ e'\ \texttt{else}\ e'''}$$

Normal Form

- Stated formally, if $(A, \rightarrow)$ is an abstract rewriting system, some $x \in A$ is in normal form if no $y \in A$ exists such that $x \rightarrow y$.
- An irreducible configuration is also called a normal form.
- Each normal form is either a final configuration or a stuck state.

Example

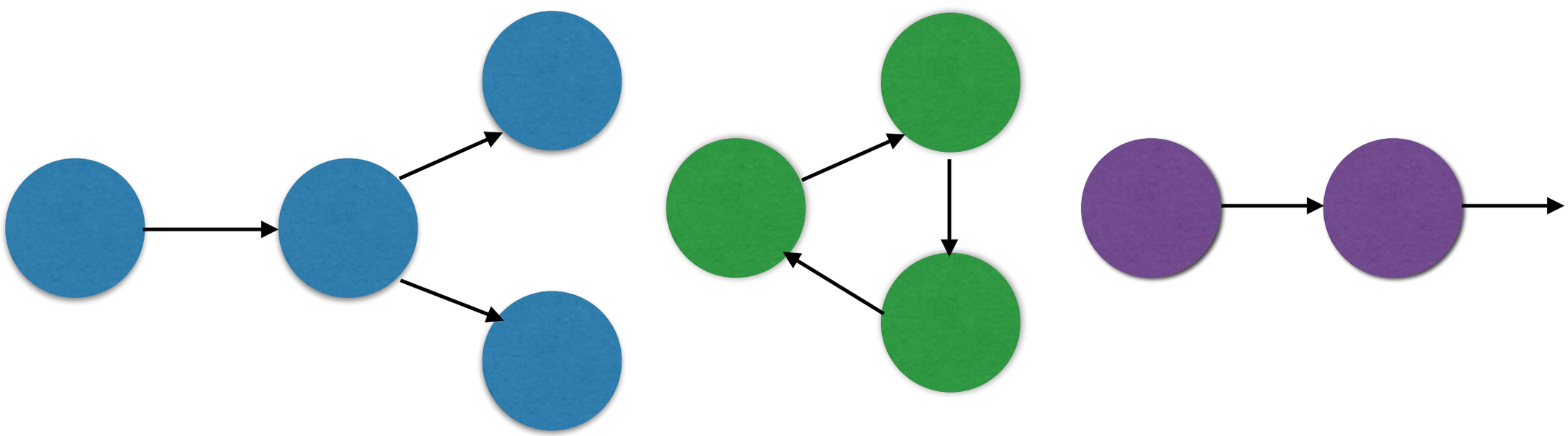
For example, using the term rewriting system with a single rule **f**(x,y)→y, the term f(f(4,2),f(3,1)) can be rewritten as follows:

$$\mathbf{f}(f(4,2),f(3,1)) \rightarrow \mathbf{f}(3,1) \rightarrow 1$$

Since no rule applies to the last term, **1**, it cannot be rewritten any further, and hence is a normal form of the term f(f(4,2),f(3,1)) with respect to this term rewriting system.

Stuck States

- Stuck States have no output according to output function.
- Stuck states include Cycling, Infinite Semantics, more than one outcome.



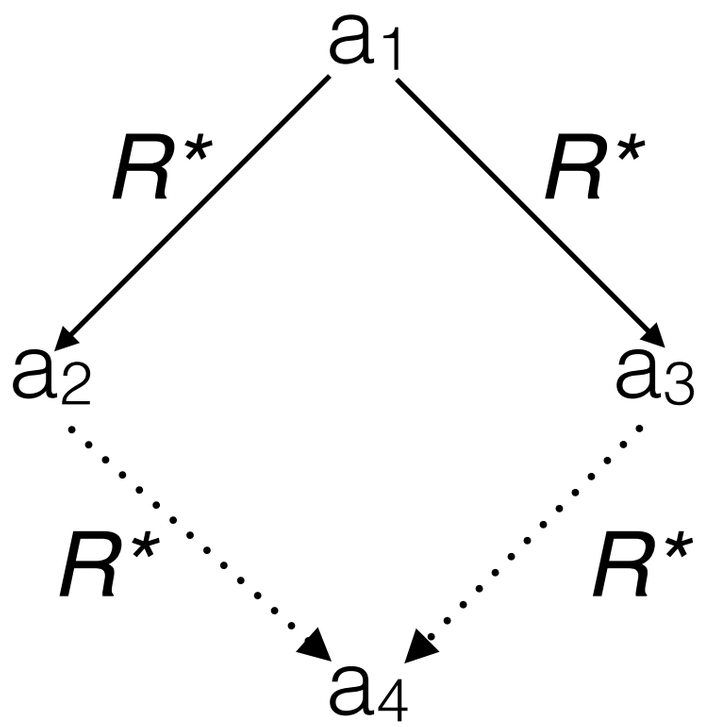
Example

One rule system:

$$g(x,y) \rightarrow g(y,x)$$

Confluence

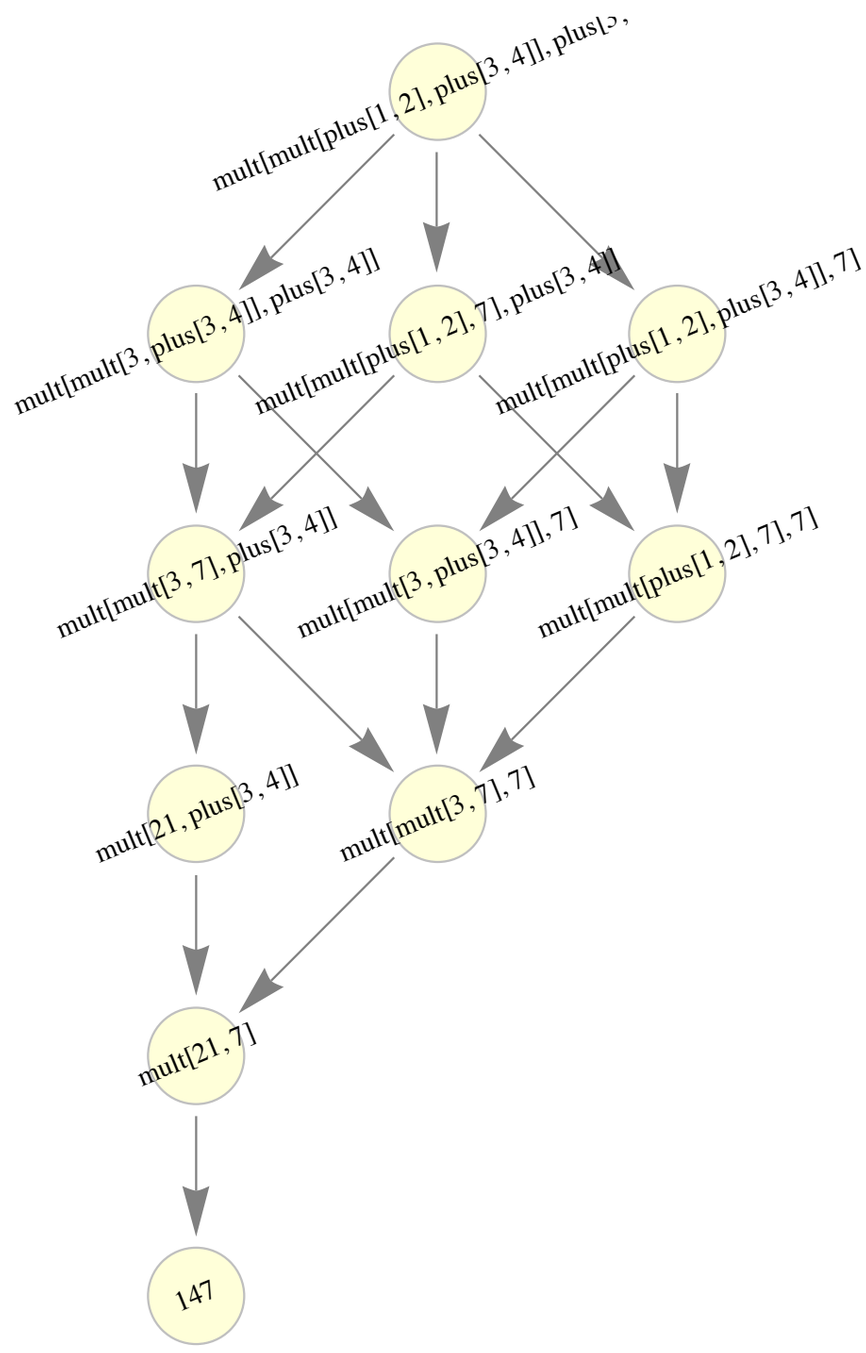
A relation $R \subseteq A \times A$ is confluent iff
$\forall a_1, a_2, a_3 \in A\colon a_1 R^* a_2 \wedge a_1 R^* a_3 \Rightarrow \exists a_4 \in A\colon a_2 R^* a_4 \wedge a_3 R^* a_4.$



Confluence

- A confluent transition relation guarantees a **unique final configuration**. Why?
- Indeed, it guarantees a **unique irreducible configuration** (normal form). Why?
- Confluence does not guarantee a **single outcome**? Why?

Confluence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mult[mult[plus[1 , 2], plus[3 , 4]], 7]
mult[mult[3 , 7], plus[3 , 4]]
mult[mult[3 , plus[3 , 4]], 7]
mult[mult[plus[1 , 2], 7], 7]
mult[21 , plus[3 , 4]]
mult[mult[3 , 7], 7]
mult[21 , 7]
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Strongly Normalising Relation

- A **strongly normalising** (or **terminating**) transition relation produces normal form via **every path**. A **weakly normalising** transition relation produces normal form via **at least one path**.

- **Well-typed** programs **don't get stuck**.

Example

One rule system:

$$g(x,y) \rightarrow x$$

Two rule system:

$$g(x,y) \rightarrow x$$

$$g(x,x) \rightarrow g(3,x)$$

Semantics Equivalence

$$\forall e \in Expr\colon e \mapsto n \Leftrightarrow e \rightsquigarrow^{*} n.$$

Type Preservation

$$\forall e, e' \in Expr\colon (\vdash e : t) \wedge e \rightsquigarrow e' \Rightarrow (\vdash e' : t).$$

Progress

$$\forall e \in Expr\colon (\vdash e : t) \Rightarrow e \in (Num \cup Bool) \vee \exists e' \in Expr\colon e \rightsquigarrow e'.$$

Termination

- Can be proved many ways.
- We show the proof based on the Energy function.
- First we define Energy function for basic expressions.

Termination

$$
\begin{aligned}
energy(v) =& 0, \\
energy(\triangle e) =& 1 + energy(e), \\
energy(e \odot e') =& 1 + energy(e) + energy(e'), \\
energy(\texttt{if } e \texttt{ then } e' \texttt{ else } e'') =& 1 + energy(e) + energy(e') + \\
& + energy(e'')
\end{aligned}
$$

energy(v)=&0,\\
energy(\triangle e) =& 1 + energy(e),\\
energy(e \odot e') =& 1 + energy(e) + energy(e'),\\
energy(\texttt{if } e \texttt{ then } e'

Termination

\forall{e} \in Expr: &energy(e) \in N
\wedge \\
\forall{e,e'} \in Expr: e \leadsto e' &
\Rightarrow energy(e) > energy(e').

$$\forall e \in Ex \qquad e) \in N \wedge$$

$$\forall e, e' \in Expr : e \rightsquigarrow e' \Rightarrow energy(e) > energy(e').$$

Determinism

- A programming language is deterministic if there is exactly one possible outcome for any pair of program and inputs.

\forall{v,v'} \in (&Num \cup Bool),
\forall{e} \in Expr: \\
e \leadsto^* v & \wedge e \leadsto^* v'
\Rightarrow v = v'.

$$\forall v, v' \in (Num \cup Bool), \forall e \in Expr :$$
$$e \rightsquigarrow^* v \wedge e \rightsquigarrow^* v' \Rightarrow v = v'.$$