AE4M33RZN, Fuzzy logic: Fuzzy description logic

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Crisp description logic

Our treatment of fuzzy description logic is based on a family of crisp description logic $\mathcal{SHIF(D)}$ [Baader, 2003]:

- $\mathcal{AL}$
  - atomic negation
  - concept intersection
  - universal restrictions
  - limited existential quantification
- $\mathcal{C}$ = full concept negation
- $\mathcal{S}$ = $\mathcal{ALC}$ + transitive roles
- $\mathcal{H}$ = role hierarchies
- $\mathcal{I}$ = inverse properties
- $\mathcal{F}$
  - concept intersection
  - universal restrictions
  - limited existential quantification
  - role restriction
- $\mathcal{D}$ = data types

$\mathcal{SHIF}$ concepts

Let A and R be the sets of *atomic concepts* and *atomic roles*.

Concept constructors

$$C, D := \top \mid \bot \qquad \text{top and bottom concepts} \tag{1}$$
$$\mid A \qquad \text{atomic concept} \tag{2}$$
$$\mid \neg C \qquad \text{concept negation} \tag{3}$$
$$\mid C \sqcap D \qquad \text{intersection} \tag{4}$$
$$\mid C \sqcup D \qquad \text{concept union} \tag{5}$$
$$\mid \forall R \cdot C \qquad \text{full universal quantification} \tag{6}$$
$$\mid \exists R \cdot C \qquad \text{full existential quantification} \tag{7}$$

Crisp description logic ontology

Ontology consists of $\mathscr{A}Box$ and $\mathscr{T}Box$. We use the set of individuals I:

$\mathscr{A}Box$ (Assertion Box)

Contains *concept assertions* $\langle i \in I : \mathsf{C} \rangle$
and *role assertions* $\langle (i, j \in I) : \mathsf{R} \rangle$.

$\mathscr{T}Box$ (Terminology Box)

Contains *general concept inclusion* (GCI) axioms $\langle \mathsf{C} \sqsubseteq \mathsf{D} \rangle$ and role axioms (role hierarchy $\langle \mathsf{R}_1 \sqsubseteq \mathsf{R}_2 \rangle$, transitivity, ...).

Crisp description logic interpretation

Interpretation $\mathscr{I}$ is a tuple $(\Delta^{\mathscr{I}}, \cdot^{\mathscr{I}})$ (*interpretation domain*, *interpretation function*), which maps

an individual to domain object	$\boldsymbol{i}^{\mathscr{I}} \in \Delta^{\mathscr{I}}$
an atomic concept to domain subsets	$\mathsf{C}^{\mathscr{I}} \subseteq \Delta^{\mathscr{I}}$
an atomic role to subset of domain tuples	$\mathsf{R}^{\mathscr{I}} \subseteq \Delta^{\mathscr{I}} \times \Delta^{\mathscr{I}}$

Crisp description logic interpretation

The non-atomic concepts are interpreted as follows:

non-atomic concept	its interpretation
$\top$	$\Delta^{\mathscr{I}}$
$\bot$	$\emptyset$
$\neg\, C$	$\Delta^{\mathscr{I}} \setminus C^{\mathscr{I}}$
$C \sqcap D$	$C^{\mathscr{I}} \cap D^{\mathscr{I}}$
$C \sqcup D$	$C^{\mathscr{I}} \cup D^{\mathscr{I}}$
$\forall R \cdot C$	$\{x \mid \forall y \in \Delta^{\mathscr{I}} . ((x, y) \in R^{\mathscr{I}}) \Rightarrow (y \in C^{\mathscr{I}})\}$
$\exists R \cdot C$	$\{x \mid \exists y \in \Delta^{\mathscr{I}} . ((x, y) \in R^{\mathscr{I}}) \wedge (y \in C^{\mathscr{I}})\}$

Crisp notion of truth

Axiom satisfaction

axiom	satisfied when
$\langle \boldsymbol{i} : \mathsf{C} \rangle$	$\boldsymbol{i}^{\mathscr{I}} \in \mathsf{C}^{\mathscr{I}}$
$\langle (\boldsymbol{i}, \boldsymbol{j}) : \mathsf{R} \rangle$	$(\boldsymbol{i}^{\mathscr{I}}, \boldsymbol{j}^{\mathscr{I}}) \in \mathsf{R}^{\mathscr{I}}$
$\langle \mathsf{C} \sqsubseteq \mathsf{D} \rangle$	$\mathsf{C}^{\mathscr{I}} \sqsubseteq \mathsf{D}^{\mathscr{I}}$
transitive(R)	$\mathsf{R}^{\mathscr{I}}$ is transitive
...	

If an interpretation satisfies τ, we write $\tau \models \mathscr{I}$.

- Interpretation $\mathscr{I}$ is a *model* of a knowledgebase $\mathscr{K}$ = $\mathscr{A}$***Box*** + $\mathscr{T}$***Box*** (or $\mathscr{I}$ *satisfies* $\mathscr{K}$) if it satisfies all its axioms.

$$\mathscr{I} \models \mathscr{K} \Leftrightarrow (\forall \tau \in \mathscr{K} \, . \; \mathscr{I} \models \tau)$$

- Axiom τ is a *logical consequence* of $\mathscr{K}$ if every model of $\mathscr{K}$ satisfies τ.

$$\mathscr{K} \models \tau \Leftrightarrow [\forall \mathscr{I} \, . \; (\mathscr{I} \models \mathscr{K}) \Rightarrow (\mathscr{I} \models \tau)]$$

- Concept C is *satisfiable* if there is an interpretation $\mathscr{I}$, where the C has at least 1 individual.

$$\exists i, \mathscr{I} \, . \; \mathscr{I} \models < i : \mathrm{C} >$$

Fuzzy description logic

Basic idea

1. Keep the the previous slides intact.
2. Add $\circ$ below and above every operation.
3. Watch the semantic change.

Male $\sqcap$ Female $\neq \bot$

Overview

We will show the **fuzzyDL** reasoner [Bobillo and Straccia, 2008] capabilities, which extends the $\mathcal{SHIF}(\mathcal{D})$ family with fuzzy capabilities.

Concept constructors

We start with atomic concepts A. Derived concepts are on the next slide together with their interpretation. (Each concept is interpreted as a fuzzy subset of the domain.)

Fuzzy DL interpretation

Fuzzy interpretation $\mathscr{I}$ is a tuple $\Delta^{\mathscr{I}}, \cdot^{\mathscr{I}}$ which maps

an individual to a domain object	$\boldsymbol{i}^{\mathscr{I}} \in \Delta^{\mathscr{I}}$
an atomic concept to a domain subsets	$\mathsf{C}^{\mathscr{I}} \in \mathbb{F}(\Delta^{\mathscr{I}})$
an atomic role to a relation on the domain	$\mathsf{R}^{\mathscr{I}} \in \mathbb{F}(\Delta^{\mathscr{I}} \times \Delta^{\mathscr{I}})$

$\mathsf{C}, \mathsf{D} :=$	interpretation of x
$\bot$	$\mathbf{0}$
$\top$	$\mathbf{1}$
A	$A^{\mathscr{I}}(x)$
$\neg\, \mathsf{C}$	$\underset{S}{\neg}\, \mathsf{C}^{\mathscr{I}}(x)$
$\mathsf{C} \underset{S}{\sqcap} \mathsf{D}$	$\mathsf{C}^{\mathscr{I}}(x) \underset{S}{\wedge} \mathsf{D}^{\mathscr{I}}(x)$
$\mathsf{C} \underset{L}{\sqcap} \mathsf{D}$	$\mathsf{C}^{\mathscr{I}}(x) \underset{L}{\wedge} \mathsf{D}^{\mathscr{I}}(x)$
$\mathsf{C} \overset{S}{\sqcup} \mathsf{D}$	$\mathsf{C}^{\mathscr{I}}(x) \overset{S}{\vee} \mathsf{D}^{\mathscr{I}}(x)$
$\mathsf{C} \overset{L}{\sqcup} \mathsf{D}$	$\mathsf{C}^{\mathscr{I}}(x) \overset{L}{\vee} \mathsf{D}^{\mathscr{I}}(x)$
$\mathsf{C} \overset{R}{\underset{S}{\mapsto}} \mathsf{D}$	$\mathsf{C}^{\mathscr{I}}(x) \overset{R}{\underset{S}{\Rrightarrow}} \mathsf{D}^{\mathscr{I}}(x)$
$\mathsf{C} \overset{R}{\underset{L}{\mapsto}} \mathsf{D}$	$\mathsf{C}^{\mathscr{I}}(x) \overset{R}{\underset{L}{\Rrightarrow}} \mathsf{D}^{\mathscr{I}}(x)$
$\mathsf{C} \overset{S}{\underset{S}{\mapsto}} \mathsf{D}$	$\mathsf{C}^{\mathscr{I}}(x) \overset{S}{\underset{S}{\Rrightarrow}} \mathsf{D}^{\mathscr{I}}(x)$

$\mathsf{C}, \mathsf{D} :=$	interpretation of x
$\exists \mathsf{R} \cdot \mathsf{C}$	$\sup_y \mathsf{R}^{\mathcal{I}}(x, y) \mathbin{\underset{\circ}{\wedge}} \mathsf{C}^{\mathcal{I}}(y)$
$\forall \mathsf{R} \cdot \mathsf{C}$	$\inf_y \mathsf{R}^{\mathcal{I}}(x, y) \mathbin{\overset{\circ}{\underset{\circ}{\Rightarrow}}} \mathsf{C}^{\mathcal{I}}(y)$
$(n\ \mathsf{C})$	$n \cdot \mathsf{C}(x)$
$mod(\mathsf{C})$	$mod(\mathsf{C}^{\mathcal{I}}(x))$
$w_1\mathsf{C}_1 + \ldots + w_k\mathsf{C}_k$	$w_1\mathsf{C}_1^{\mathcal{I}}(x) + \ldots + w_k\mathsf{C}_k^{\mathcal{I}}(x)$
$\mathsf{C} \lesseqqgtr n$	$\begin{cases} \mathsf{C}^{\mathcal{I}}(x) & \mathsf{C}^{\mathcal{I}}(x) \lesseqqgtr n \\ 0 & \text{otherwise} \end{cases}$

Modifiers

Modifier is a function that alters the membership function.

Example

Linear modifier of degree c is

$$a = \frac{c}{c+1}$$
$$b = \frac{1}{c+1}$$

Fuzzy DL ontology

Ontology consists of axioms (denoted τ) in $\mathscr{A}$***Box*** and $\mathscr{T}$***Box***:

$\mathscr{A}$*Box* (Assertion Box)

Contains ***concept assertions*** $\langle \boldsymbol{i} \in \boldsymbol{I} : \mathsf{C} \,|\, \alpha \rangle$
and ***role assertions*** $\langle (\boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{I}) : \mathsf{R} \,|\, \alpha \rangle$.

$\mathscr{T}$*Box* (Terminology Box)

GCI axioms $\langle \mathsf{C} \sqsubseteq \mathsf{D} \,|\, \alpha \rangle$ state that “C is D **at least** by α”.
Besides GCI, there are role hierarchy axioms $\langle \mathsf{R}_1 \sqsubseteq \mathsf{R}_2 \rangle$, transitivity axioms and definitions of inverse relations.

Fuzzy truth

Fuzzy axioms

axiom	satisfied if
$\langle i : C \mid \alpha \rangle$	$C^{\mathscr{I}}(i^{\mathscr{I}}) \geq \alpha$
$\langle (i,j) : R \mid \alpha \rangle$	$R^{\mathscr{I}}(i^{\mathscr{I}}, j^{\mathscr{I}}) \geq \alpha$
$\langle C \sqsubseteq D \mid \alpha \rangle$	$C^{\mathscr{I}} \overset{\circ}{\underset{\circ}{\subseteq}} D^{\mathscr{I}} \geq \alpha$
$\langle R_1 \sqsubseteq R_2 \rangle$	$R_1^{\mathscr{I}} \subseteq R_2^{\mathscr{I}}$
$\langle \text{transitive } R \rangle$	$R^{\mathscr{I}}$ is $\circ$-transitive
$\langle R_1 = R_2^{-1} \rangle$	$R_1^{\mathscr{I}} = (R_2^{\mathscr{I}})^{-1}$

Using these definitions,
the notions of *logical consequence* and *satisfiability*
(of both concepts and axioms)
remains the same.
More on slide 307.

What can you ask the reasoner?

Best/Worst Degree Bound

What is the minimal degree of an axiom that $\mathscr{K}$ ensures?

$$\mathrm{glb}(\mathscr{K}, \tau) = \mathbf{sup}\{\alpha \mid \mathscr{K} \models \langle \tau \geq \alpha \rangle\}$$
$$\mathrm{lub}(\mathscr{K}, \tau) = \mathbf{inf}\{\alpha \mid \mathscr{K} \models \langle \tau \leq \alpha \rangle\}$$

where τ is an axiom of type $\langle i : \mathsf{C} \rangle$ or $\langle (i, j) : \mathsf{R} \rangle$ or $\langle \mathsf{C} \sqsubseteq \mathsf{D} \rangle$.

- From an empty $\mathscr{K}$, you cannot infer anything and therefore $\mathrm{glb}(\mathscr{K}, \tau) = 0$ and $\mathrm{lub}(\mathscr{K}, \tau) = 1$ (if using atomic concepts only). Only by adding new axioms into $\mathscr{K}$, the bounds "tighten up".
- What happens if $\mathrm{glb}(\mathscr{K}, \tau) \geq \mathrm{lub}(\mathscr{K}, \tau)$ for some axiom τ?

What can you ask the reasoner?

Best Satisfiability Bound

What is the maximal degree of satisfiability of C?

$$\mathrm{glb}(\mathscr{K}, \mathsf{C}) = \sup_{\mathscr{I}} \sup_{x \in \Delta} \{\mathsf{C}^{\mathscr{I}}(x) \mid \mathscr{I} \models \mathscr{K}\} \,.$$

This is a generalization of *concept satisfiability*.

Homework

Next time we will see a reasoning algorithm for fuzzy DL. Please read [Straccia and Bobillo, 2008]:

Basic idea of the fuzzyDL solver:

Straccia, Umberto and Fernando Bobillo. “**Mixed integer programming, general concept inclusions and fuzzy description logics.**”
Mathware & Soft Computing 14, no. 3 (2008): 247-259.

Where can you find the article? Google scholar is a place to start.

Bibliography

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