

## Languages, grammars, automata

Czech instant sources:
[1] Prof. Marie Demlová: A4B01JAG
http://math.feld.cvut.cz/demlova/teaching/jag/predn_jag.html
Pages 1-27, in PAL, you may wish to skip: Proofs, chapters 2.4, 2.6, 2.8.
[2] I. Černá, M. Křetínský, A. Kučera: Automaty a formální jazyky I
http://is.muni.cz/do/1499/el/estud/fi/js06/ib005/Formalni_jazyky_a_automaty_I.pdf
Chapters 1 and 2, skip same parts as in [1].
English sources:
[3] B. Melichar, J. Holub, T. Polcar: Text Search Algorithms
http://cw.felk.cvut.cz/lib/exe/fetch.php/courses/a4m33pal/melichar-tsa-lectures-1.pdf
Chapters 1.4 and 1.5 , it is probably too short, there is nothing to skip.
[4] J. E. Hopcroft, R. Motwani, J. D. Ullman: Introduction to Automata Theory folow the link at http://cw.felk.cvut.cz/doku.php/courses/a4m33pal/literatura_odkazy Chapters 1., 2., 3., there is a lot to skip, consult the teacher preferably.

For more references see PAL links page
http://cw.felk.cvut.cz/doku.php/courses/a4m33pal/literatura_odkazy

Deterministic Finite Automaton (DFA) Nondeterministic Finite Automaton (NFA)

Both DFA nd NFA consist of:
Finite input alphabet $\Sigma$.
Finite set of internal states $Q$.
One starting state $q_{0} \in Q$.
Nonempty set of accept states $F \subseteq Q$.
Transition function $\delta$.
DFA transition function is $\delta: Q \times \Sigma \rightarrow Q$.


DFA is always in one of its states.
DFA transits from current state to another state depending on the current input symbol.
NFA transition function is $\delta: \mathrm{Q} \times \Sigma \rightarrow P(Q) \quad(P(Q)$ is powerset of Q , set of all subsets of Q ) NFA is always (simultaneously) in a set of any number of its states.
NFA transits from a state to a set of states depending on the current input symbol.

NFA $\mathbf{A}_{1}$, its transition diagram and its transition table


|  | $a$ | $b$ | $c$ |
| :---: | :---: | :---: | :---: |
| 0 | 1 | 2 |  |
| 1 |  | 3,4 |  |
| 2 | 4,5 |  |  |
| 3 | 6 |  | 0 |
| 4 |  |  | 6,7,8 |
| 5 |  | 8 |  |
| 6 | 0 |  |  |
| 7 | 6 | 6 |  |
| 8 | 7 | 7 |  |

NFA $A_{1}$ processing input word abcba



NFA $\mathrm{A}_{1}$ has processed word abcba and went through read symbols and respective sets(!) of states
$\{0\} \rightarrow \mathrm{a} \rightarrow\{1\} \rightarrow \mathrm{b} \rightarrow\{3,4\} \rightarrow \mathrm{c} \rightarrow$
$\rightarrow\{0,6,7,8\} \rightarrow b \rightarrow\{2,6,7\} \rightarrow a \rightarrow$
$\rightarrow\{0,4,5,6\}$.

## NFA simulation without transform to DFA

Each of current states is occupied by one token.
Read an input symbol and move tokens accordingly.
If token has more possibilities it will split into two or more tokens, if token has no possibility it will leave the board, uhm, the transition diagram.


## NFA simulation without transform to DFA

## Idea:

Register all states to which you have just arrived. In the next step read the input symbol $x$ and move SIMULTANEOUSLY to ALL states to which you can get from ALL current states along transitions marked by x .

```
    Input: NFA, text in array t
SetOfStates S = {q0}, S_tmp;
i = 1;
while ((i <= t.length) && (!S.isEmpty())) {
    S_tmp = Set.emptySet();
    for (q in S) // for each state in S
            S_tmp.union(delta(q, t[i]));
    S = S_tmp;
    i++;
}
return S.containsFinalState(); // true or false
```

Generating DFA $\mathbf{A}_{\mathbf{2}}$ equivalent to NFA $\mathbf{A}_{\mathbf{1}}$ using transition tables
Data
Each state of DFA is a subset of states of NFA
Start state of DFA is a one element set containing just start state of NFA.
A state of DFA is accept state iff it contains at least one accept state of NFA.
Construction
Create start state of DFA and corresponding first line of its transition table (TT).
For each state Q of DFA not yet processed do \{
Decompose Q into its constituent states Q1, ..., Qk of NFA
For each symbol $x$ of alphabet do \{
$S=$ union of all references in NFA table at positions [Q1] [x], ... [Qk][x]
if ( S is not among states of DFA yet)
add $S$ to states of DFA and add corresponding line to TT of DFA
\}
Mark Q as processed
\}
// Remember, empty set is also a set ot states, it can be easily a state of DFA

Generating DFA $\mathbf{A}_{2}$ equivalent to NFA $\mathbf{A}_{1}$


Copy start state
$\mathrm{A}_{2}$

|  | $a$ |  | $\boldsymbol{b}$ |
| :---: | :---: | :---: | :---: |
| $c$ |  |  |  |
| 0 | 1 | 2 |  |
| $\ldots$ |  |  |  |
|  |  |  |  |

Generating DFA $\mathbf{A}_{2}$ equivalent to NFA $\mathbf{A}_{1}$


Add new state(s)

|  | a | b | c |
| :---: | :---: | :---: | :---: |
| 0 | 1 | 2 |  |
| 1 |  | 34 |  |
| ... |  |  |  |

Generating DFA $\mathbf{A}_{2}$ equivalent to NFA $\mathbf{A}_{1}$


Add new state(s)

|  | $a$ | $b$ | c |
| :---: | :---: | :---: | :---: |
| 0 | 1 | 2 |  |
| 1 |  | 34 |  |
| 2 | 45 |  |  |
| ... |  |  |  |

Generating DFA $\mathbf{A}_{\mathbf{2}}$ equivalent to NFA $\mathbf{A}_{1}$


## Add new state(s)

|  | $a$ | $b$ | $c$ |
| ---: | :---: | :---: | :---: |
| 0 | 1 | 2 |  |
| 1 |  | 34 |  |
| 2 |  | $F$ |  |
| 34 |  |  |  |
|  | 6 |  | 0678 |
|  |  |  |  |
|  |  |  |  |

## Note:

In the example we add the empty set to the table at the very end of the process just to keep the table uncluttered...

Generating DFA $\mathbf{A}_{\mathbf{2}}$ equivalent to NFA $\mathbf{A}_{\mathbf{1}}$


Add new state(s)
$\mathrm{A}_{2}$

|  | $a$ | $b$ | c |
| :---: | :---: | :---: | :---: |
| 0 | 1 | 2 |  |
| 1 |  | 34 |  |
| 2 | 45 |  |  |
| 34 | 6 |  | 0678 |
| 45 |  | 8 | 678 |
| ... |  |  |  |

Generating DFA $\mathbf{A}_{2}$ equivalent to NFA $\mathbf{A}_{1}$

## ... after few more iterations...

|  | $a$ | b | c |
| :---: | :---: | :---: | :---: |
| 0 | 1 | 2 |  |
| 1 |  | 3,4 |  |
| 2 | 4,5 |  |  |
| 3 | 6 |  | 0 |
| 4 |  |  | 6,7,8 |
| 5 |  | 8 |  |
| 6 | 0 |  |  |
| 7 | 6 | 6 |  |
| 8 | 7 | 7 |  |


$\mathrm{A}_{\mathbf{2}}$
Add new state(s)

|  | $a$ | $b$ | c |
| :---: | :---: | :---: | :---: |
| 0 | 1 | 2 |  |
| 1 |  | 34 |  |
| 2 | 45 |  |  |
| 34 | 6 |  | 0678 |
| 45 |  | 8 | 678 |
| 6 | 0 |  |  |
| 0678 | 0167 | 23467 |  |
| 8 | 7 | 7 |  |
| 678 | 067 | 67 |  |
| 0167 | 016 | 2346 |  |
| 23467 | 0456 | 6 | 0678 |
| 7 | 6 | 6 |  |
| 067 |  |  |  |
| ... |  |  |  |

DFA $\mathbf{A}_{\mathbf{2}}$ equivalent to NFA $\mathbf{A}_{1}$
..FINISHED!

|  | a | b | $c$ |
| :---: | :---: | :---: | :---: |
| 0 | 1 | 2 |  |
| 1 |  | 3,4 |  |
| 2 | 4,5 |  |  |
| 3 | 6 |  | 0 |
| 4 |  |  | 6,7,8 |
| 5 |  | 8 |  |
| 6 | 0 |  |  |
| 7 | 6 | 6 |  |
| 8 | 7 | 7 |  |



| $a$ | b | c |
| :---: | :---: | :---: |
| 1 | 2 | n |
| n | 34 | n |
| 45 | n | n |
| 6 | n | 0678 |
| n | 8 | 678 |
| 0 | n | n |
| 0167 | 23467 | n |
| 7 | 7 | n |
| 067 | 67 | n |
| 016 | 2346 | n |
| 0456 | 6 | 0678 |
| 6 | 6 | n |
| 016 | 2346 | n |
| 06 | 6 | n |
| 01 | 234 | n |
| 0456 | n | 0678 |
| 01 | 28 | 678 |
| 01 | 2 | n |
| 1 | 234 | n |
| 456 | n | 0678 |
| 457 | 7 | n |
| 0 | 8 | 678 |
| 6 | 68 | 678 |
| 07 | 7 | n |
| 16 | 26 | n |
| 0 | 34 | n |
| 045 | n | n |
| 1 | 28 | 678 |
| n | n | n |

Naïve approach

1. Align pattern with the beginning of text.
2. While corresponding symbols of pattern and text match each other move forward by one symbol in pattern.
3. When symbol mismatch occurs shift pattern forward by one symbol, reset position in the pattern to the beginning of pattern and go to 2 .
4. When the end of pattern is passed report success, reset position in the pattern to its beginning and go to 2 .
5. When the end of text is reached stop.

Might be both efficient and not

## Start


pattern $\mathrm{a}|\mathrm{b}| \mathrm{c} \mid \mathrm{x}$

## Pattern shift


pattern

after a while:

pattern

| $a$ | $b$ | $c$ | $x$ |
| :--- | :--- | :--- | :--- |

match
mismatch

Alphabet: Finite set of symbols.
Text: Sequence of symbols of the alphabet.
Pattern: Sequence of symbols of the same alphabet, pattern occurence is to be detected in the text

Text is often fixed or seldom changed, pattern typically varies (looking for different words in the same document), pattern is often significantly shorter than the text.

## Notation

Alphabet: $\Sigma$
Symbols in the text: $t_{1}, t_{2}, \ldots t_{n}$
Symbols in the pattern: $p_{1}, p_{2}, \ldots p_{m}$
Holds $m \leq n$, usually $m \ll n$

## Example

Text: ...task is very simple but it is used very freq... Pattern: simple

## Boyer - Moore

The idea:
Align the pattern with the text and check and start matching backwards from the end of the pattern. When a mismatch occurs there is a chance that a pattern may be shifted forward by many positions sometimes by the whole pattern length.

## Ideal case



The longer is the pattern the more effective is the search. (The bigger the data the faster the algorithm, quite an unusual situation...)

Mismatch at the last position of the pattern.
Bad Character Shift table (BCS)
When the last symbol of pattern (y) is mismatched with symbol $\mathbf{x}$ in the text shift the pattern to the right to match the first occurrence (from the end) of $\mathbf{x}$ in the pattern with $\mathbf{x}$ in the text.
When the pattern does not contain $\mathbf{x}$ shift it by its whole length.
BCS is indexed by all symbols of alphabet. For each symbol it contains the symbol's minimum distance from the end of the pattern.

## Example



## Mismatch after partial match at the end of the pattern.

When a suffix $S$ of the pattern matches the text and the symbol $\mathbf{x}$ immediately preceding $S$ mismatches the text then there are three cases:

1. The suffix $S$ occurs more times in the pattern and the other occurrence is not immediately precede by $x$. In this case, shift the pattern so that the nearest described instance of $S$ matches the text again at the same position. That is, shift the pattern by the distance between these occurrences of suffix S .

## Example


2. There is a suffix $W$ which length does not exceed the length of $S$ and $W$ is also A prefix of the pattern. Take the longest possible W and denote its occurrence at the beginning of the pattern by Q .
Then shift the pattern by the distance between $Q$ and $W$.

## Example

Text


Shift after mismatch
$x|y| x|y| b|f| l|m| a|z| x|y| x \mid y$

Longest suffix after mismatch which is also a prefix
3. Neither case 1. nor case 2. happens. Then shift the pattern by its whole length.

## Example is unnecessary

The shift can be calculated for all three cases:
Take suffix $S$ as a separate string and align it with its original position in the pattern. Then keep shifting $S$ to to the left until one of the cases 1., 2., 3. is detected (at least 3. must happen after some time).
Register the distance between the current and the original position of $S$.
Good Suffix Shift (GS) table contains the shift values for all suffixes $S$.

## Example

Pattern | $\mathbf{A}$ | $\mathbf{D}\|\mathbf{B}\|$ |
| :--- | :--- |
| $\mathbf{A}$ | $\mathbf{C}$ |
| $\mathbf{B}$ | $\mathbf{B}$ |
| $\mathbf{A}$ | $\mathbf{C}$ |
| $\mathbf{B}$ | $\mathbf{B}$ |
| $\mathbf{A}$ | $\mathbf{A}$ |

Pattern length: 10
Positions indexed from 1, 0 represents shift after complete match.

Apply case 2. after complete match

GS

| position | mismatches | suffix | shift |
| :---: | :---: | :---: | :---: |
| 9 | B | A | 9 |
| 8 | C | BA | 6 |
| 7 | A | CBA | 9 |
| 6 | B | ACBA | 9 |
| 5 | C | BACBA | 3 |
| 4 | A | CBACBA | 9 |
| 3 | B | ACBACBA | 9 |
| 2 | D | BACBACBA | 9 |
| 1 | A | DBACBACBA | 10 |
| 0 | - | ADBACBACBA | 9 |

Example
Pattern $\square$
BCS
GS

| $\mathbf{P}$ | $\mathbf{O}$ | $\mathbf{V}$ | $\mathbf{A}$ | $\mathbf{L}$ | $\mathbf{O}$ | $\mathbf{V}$ | $\mathbf{A}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |


| - | A | E | K | L | N | O | P | S | T |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 9 | $\mathbf{V}$ |  |  |  |  |  |  |  |  |
| $\mathbf{y}$ | $\mathbf{9}$ | 9 | 4 | 9 | 3 | 8 | 9 | 9 | 2 |

Search progress

$$
\text { BCS[P] }==8
$$

$$
\text { GS[5] }==4
$$

$$
\text { GS[6] }==9
$$

Text

POVALO|V|A|L

$$
\begin{aligned}
& 0123456789 \\
& \begin{array}{|l|l|l|l|l|l|l|l|l|}
\hline 9 & 9 & 9 & 9 & 9 & 4 & 9 & 9 & 9
\end{array} \text { - } \\
& \text { POVALOVAL }
\end{aligned}
$$

NFA $A_{3}$ which accepts just a single word $p_{1} p_{2} p_{3} p_{4}$.
$\mathrm{A}_{3}$
$\longrightarrow(0) \xrightarrow{p_{1}}$ (1)
$\xrightarrow{p_{2}}$ (2)
$\xrightarrow{p_{3}}(3)$
$\xrightarrow{\mathrm{p}_{4}}$ (4)

NFA $A_{4}$ which accepts each word with suffix $p_{1} p_{2} p_{3} p_{4}$ with its transition table.
$\mathrm{A}_{4}$


|  | $p_{1}$ | $p_{2}$ | $p_{3}$ | $p_{4}$ | $z$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0,1 | 0 | 0 | 0 | 0 |
| 1 |  | 2 |  |  |  |
| 2 |  |  | 3 |  |  |
| 3 |  |  |  | 4 |  |
| 4 |  |  |  |  |  |
| $z \in \Sigma-\{p 1, p 2, p 3, p 4\}$ |  |  |  |  |  |

## repeated

NFA $A_{4}$ which accepts each word with suffix $p_{1} p_{2} p_{3} p_{4}$ and its transition table.


|  | $p_{1}$ | $p_{2}$ | $p_{3}$ | $p_{4}$ | $z$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0,1 | 0 | 0 | 0 | 0 |
| 1 |  | 2 |  |  |  |
| 2 |  |  | 3 |  |  |
| 3 |  |  |  | 4 |  |
| 4 |  |  |  |  |  |
| $z \in \Sigma-\{p 1, p 2, p 3, p 4\}$ |  |  |  |  |  |

## equivalently

## DFA $\mathrm{A}_{5}$ is a deterministic equivalent of NFA $\mathrm{A}_{4}$.



|  | $p_{1}$ | $p_{2}$ |  |  | $z$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 01 | 0 | 0 | 0 | 0 |
| 01 | 01 | 02 | 0 | 0 | 0 |
| 02 | 01 | 0 | 03 | 0 | 0 |
| 03 | 01 | 0 | 0 | 04 | 0 |
| 04 | 01 | 0 | 0 | 0 | 0 |

example
NFA $A_{6}$ which accepts each word with suffix abba with its transition table



DFA $\mathrm{A}_{7}$ is a deterministic equivalent of NFA $\mathrm{A}_{6}$. It also accepts each word with suffix abba.


|  | $a$ |  | $b$ |
| :---: | :---: | :---: | :---: |
| $z$ |  |  |  |
| 0 | 01 | 0 | 0 |
| 01 | 01 | 02 | 0 |
| 02 | 01 | 03 | 0 |
| 03 | 014 | 0 | 0 |
| 04 | 01 | 02 | 0 |
|  |  |  |  |

NFA accepting exactly one word $p_{1} p_{2} p_{3} p_{4}$.

$$
\rightarrow(0) \xrightarrow{p_{1}}(1) \xrightarrow{p_{2}}(2) \xrightarrow{p_{3}}(3) \xrightarrow{p_{4}}
$$

NFA accepting any word with suffix $p_{1} p_{2} p_{3} p_{4}$.

$$
\xrightarrow[(0)]{Q_{1}^{\Sigma}}(1) \xrightarrow{p_{2}}(2) \xrightarrow{p_{3}}(3) \xrightarrow{p_{4}} \text { (4) }
$$

NFA accepting any word with substring (factor) $p_{1} p_{2} p_{3} p_{4}$ anywhere in it.

$$
\xrightarrow[(0) p_{1}]{Q^{\Sigma}} \text { (1) } p_{2} \xrightarrow{p_{3}} \text { (3) } \xrightarrow{p_{4}}
$$

NFA accepting any word with substring (factor) $p_{1} p_{2} p_{3} p_{4}$ anywhere in it.


Can be used for search, but the following reduction is usual.
Text search NFA for finding pattern $P=p_{1} p_{2} p_{3} p_{4}$ in the text.

$$
\xrightarrow[(0)]{Q_{1}^{\Sigma} p_{1}}(1) \xrightarrow{p_{2}}(2) \xrightarrow{p_{3}}(3) \xrightarrow{p_{4}}(4)
$$

NFA stops when pattern is found.

Want to know the position of the pattern in the text? Equip the transitions with a counter.


## Example

NFA accepting any word with subsequence $p_{1} p_{2} p_{3} p_{4}$ anywhere in it.


## Example

NFA accepting any word with subsequence $p_{1} p_{2} p_{3} p_{4}$ anywhere in it, one symbol in the sequence may be altered.


Alternatively: NFA accepting any word containing a subsequence $\mathbf{Q}$ which Hamming distance from $p_{1} p_{2} p_{3} p_{4}$ is at most 1.

Search NFA can search for more than one pattern simultaneously.
The number of patterns can be
finite -- this leads to dictionary automaton (we will meet them later)
or infinite -- this leads to regular language.

## Chomsky language hierarchy remainder

## Grammar Language

Type-0 Recursively enumerable
Type-1 Context-sensitive
Type-2 Context-free
Type-3 Regular

## Automaton

Turing machine
Linear-bounded non-deterministic Turing machine
Non-deterministic pushdown automaton Finite state automaton (NFA or DFA)

Only regular languages can be processed by NFA/DFA. More complex languages cannot. For example any language containing well-formed parentheses is context-free and not regular and cannot be recognized by NFA/DFA.

## Operations on regular languages

Let $L_{1}$ and $L_{2}$ be any languages. Then
$L_{1} \cup L_{2}$ is union of $L_{1}$ and $L_{2}$. It is a set of all words which are in $L_{1}$ or $L_{2}$. $L_{1} . L_{2}$ is concatenation of $L_{1}$ and $L_{2}$. It is a set of all words $w$ for which holds $w=w_{1} w_{2}$ (concatenation of words $w_{1}$ and $w_{2}$ ), where $w_{1} \in L_{1}$ and $w_{2} \in L_{2}$.
$L_{1}{ }^{*}$ is Kleene star or Kleene closure of language $L_{1}$. It is set of all words which are concatenations of any number (incl. zero) of any words of $L_{1}$ in any order.

## Closure

Whenever $L_{1}$ and $L_{2}$ are regular languages then $L_{1} \cup L_{2}, L_{1} \cdot L_{2}, L_{1}{ }^{*}$ are regular languages too.

## Example

$L_{1}=\{001,0001,00001, \ldots\}, L_{2}=\{110,1110,11110, \ldots\}$.
$L_{1} \cup L_{2}=\{001,110,0001,1110,0001,1110, \ldots\}$
$L_{1} \cdot L_{2}=\{001110,0011110,00111110, \ldots, 0001110,00011110,000111110, \ldots\}$
$\mathrm{L}_{1}{ }^{*}=\{\varepsilon, 001,001001,001001001, \ldots 0010001,00100010001, \ldots$
..., 00100001, 001000001, ... \} // this one is not easy to list nicely ... or is it?

## Regular expressions defined recursively

Symbol $\varepsilon$ is regular expression.
Each symbol of alphabet $\Sigma$ is regular expression.
Whenever $e_{1}$ and $e_{2}$ are regular expressions also strings $\left(e_{1}\right), e_{1}+e_{2}, e_{1} e_{2},\left(e_{1}\right)^{*}$ are regular expressions.

## Languages represented by regular expressions (RE) defined recursively

$\mathrm{RE} \varepsilon$ represents language containing only empty string
RE $x$, where $x \in \Sigma$, represents language $\{x\}$.
Let RE's $\mathrm{e}_{1}$ and $\mathrm{e}_{2}$ represent languages $\mathrm{L}_{1}$ and $\mathrm{L}_{2}$. Then $R E\left(e_{1}\right)$ represents $L_{1}, R E e_{1}+e_{2}$ represents $L_{1} \cup L_{2}, R E e_{1} e_{2}$, represents $L_{1} \cdot L_{2}$, RE $\left(e_{1}\right)^{*}$ represents $L_{1}{ }^{*}$.

## Examples

$0+1(0+1)^{*}$ all integers in binary without leading 0 's
$0 .(0+1)^{*} 1 \quad$ all finite binary fractions $\in(0,1)$ without trailing 0 's
$((0+1)(0+1+2+3+4+5+6+7+8+9)+2(0+1+2+3)):(0+1+2+3+4+5)(0+1+2+3+4+5+6+7+8+9)$
all 1440 day's times in format hh:mm from 00:00 to 23:59
(mon+(wedne+t(ue+hur))s+fri+s(atur+un))day
English names of days in the week
$(1+2+3+4+5+6+7+8+9)(0+1+2+3+4+5+6+7+8+9)^{*}((2+7) 5+(5+0) 0)$
all decimal integers $\geq 100$ divisible by 25

## Convert regular expression to NFA

Input: Regular expression R containing $n$ characters of the given alphabet. Output: NFA recognizing language $L(R)$ described by $R$.

Create start state S
for each $k(1 \leq k \leq n)\{$
assign index $k$ to the $k$-th character in R
// this makes all characters in R unique: $\mathrm{c}[1], \mathrm{c}[2], \ldots, \mathrm{c}[n]$.
create state $\mathrm{S}[k] \quad / / \mathrm{S}[k]$ corresponds directly to $\mathrm{c}[k]$
\}
for each $k(1 \leq k \leq n)\{$
if $c[k]$ can be the first character in some string described by $R$ then create transition $S \rightarrow S[k]$ labeled by $c[k]$ with index stripped off
if $c[k]$ can be the last character in some string described by $R$ then mark $\mathrm{S}[\mathrm{k}]$ as final state
for each $p(1 \leq p \leq n)$
if ( $c[k]$ can follow immediately after $c[p]$ in some string described by R ) then create transition $\mathrm{S}[p] \rightarrow \mathrm{S}[k]$ labeled by $c[k]$ with index stripped off
$\underbrace{R=a^{*} b\left(c+a^{*} b\right)^{*} b+c}_{\text {Regular expression }}$
$\quad$ Add indices:
$R=a_{1}{ }^{*} b_{2}\left(c_{3}+a_{4}{ }^{*} b_{5}\right)^{*} b_{6}+c_{7}$

NFA accepts L(R)


NFA searches the text for any occurence of any word of $L(R)$
$R=a^{*} b\left(c+a^{*} b\right)^{*} b+c$

The only difference from the NFA accepting R


## Bonus

To find a subsequence representing a word $\in L(R)$, where $R$ is a regular expression, do the following:

Create NFA acepting $L(R)$
Add self loops to the states of NFA:

1. Self loop labeled by $\Sigma$ (whole alphabet) at the start state.
2. Self loop labeled $\Sigma-\{x\}$ at each state which outgoing transition(s) are labeled by single $x \in \Sigma$.
// serves as an "optimized" wait loop
3. Self loop labeled by $\Sigma$ at each state which outgoing transition(s) are labeled by more than single symbol from $\Sigma$.// serves as an "usual" wait loop
4. No self loop to all other states. // which have no outgoing loop, final ones

## Bonus

NFA searches the text for any occurence of any subsequence representing a word word of $L(R)$
$R=a b+(a b c b+c c)^{*} a$


Transforming NFA which searches text for an occurence of a word of a given regular language into the equivalent DFA might take exponential space and thus also exponential time. Not always, but sometimes yes:

Consider regular expression $\mathbf{R}=\mathbf{a}(\mathbf{a}+\mathbf{b})(\mathbf{a}+\mathbf{b}) \ldots(\mathbf{a}+\mathbf{b})$ over alphabet $\{a, b\}$.

## Text search NFA1 for R

NFA1


## Mystery

Text search NFA2 for R, why not this one?
NFA2



## Search the text for more than just exact match

## NFA with $\varepsilon$-transitions

The transition from one state to another can be performed without reading any input symbol. Such transition is labeled by symbol $\varepsilon$.

## $\varepsilon$-closure

Symbol $\varepsilon$-CLOSURE $(p)$ denotes the set of all states $q$, which can be reached from $p$ using only $\varepsilon$-transitions.
By definition let $\varepsilon-\operatorname{CLOSURE}(p)=\{p\}$, when there is no $\varepsilon$-transition out from $p$.

$$
\begin{aligned}
\varepsilon-\operatorname{CLOSURE}(0) & =\{2,3,4\} \\
\varepsilon-\operatorname{CLOSURE}(1) & =\{1\} \\
\varepsilon-\operatorname{CLOSURE}(2) & =\{3,4\} \\
\varepsilon-\operatorname{CLOSURE}(3) & =\{3\}
\end{aligned}
$$

## Construction of equivalent NFA without $\varepsilon$-transitions

Input: NFA $A$ with some $\varepsilon$-transitions.
Output: NFA $A^{\prime}$ without $\varepsilon$-transitions.

1. $A^{\prime}=$ exact copy of $A$.
2. Remove all $\varepsilon$-transitions from $A^{\prime}$.
3. In $A^{\prime}$ for each ( $\mathrm{q}, \mathrm{a}$ ) do: add to the set $\delta(p, a)$ all such states $r$ for which holds $\quad q \in \varepsilon-\operatorname{CLOSURE}(p)$ and $\delta(q, a)=r$.
4. Add to the set of final states $F$ in $A^{\prime}$ all states $p$ for which holds $\varepsilon-\operatorname{CLOSURE}(p) \cap F \neq \varnothing$.

## easy construction




NFA for search for any unempty substring of pattern $p_{1} p_{2} p_{3} p_{4}$ over alphabet $\Sigma$.
Note the $\varepsilon$-transitions.


## Powerful trick!

Union of two or more NFA:
Create additional start state $S$ and add $\varepsilon$-transitions from $S$ to start states of all involved NFA's. Draw an example yourself!

Equivalent NFA for search for any unempty substring of pattern $p_{1} p_{2} p_{3} p_{4}$ with $\varepsilon$-transitions removed.

States 5, 9, 12 are unreachable. Transformation algorithm NFA -> DFA if applied, will neglect them.


|  | $p_{1}$ | $p_{2}$ | $p_{3}$ | $p_{4}$ | $z$ |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0,1 | 0,6 | 0,10 | 0,13 | 0 |  |  |  |  |  |  |  |  |
| 1 |  | 2 |  |  | 0 | F |  |  |  |  |  |  |  |
| 2 |  |  | 3 |  | 0 | F |  | $p_{1}$ | $p_{2}$ | $p_{3}$ | $p_{4}$ | $z$ |  |
| 3 |  |  |  | 4 | 0 | F | 0 | 0.1 | 0.6 | 0.10 | 0.13 | 0 |  |
| 4 |  |  |  |  | 0 | F | 0.1 | 0.1 | 0.2.6 | 0.10 | 0.13 | 0 | $F$ |
| 5 |  | 6 | 10 | 13 | 0 |  | 0.6 | 0.1 | 0.6 | 0.7.10 | 0.13 | 0 | $F$ |
| 6 |  |  | 7 |  | 0 | F | 0.10 | 0.1 | 0.6 | 0.10 | 0.11.13 | 0 | $F$ |
| 7 |  |  |  | 8 | 0 | F | 0.13 | 0.1 | 0.6 | 0.10 | 0.13 | 0 | $F$ |
| 8 |  |  |  |  | 0 | F | 0.2.6 | 0.1 | 0.6 | 0.3.7.10 | 0.13 | 0 | $F$ |
| 9 |  |  | 10 | 13 | 0 |  | 0.7.10 | 0.1 | 0.6 | 0.10 | 0.8.11.13 | 0 | $F$ |
| 10 |  |  |  | 11 | 0 | F | 0.11.13 | 0.1 | 0.6 | 0.10 | 0.13 | 0 | $F$ |
| 11 |  |  |  |  | 0 | F | 0.3.7.10 | 0.1 | 0.6 | 0.10 | 0.4.8.11.13 | 0 | $F$ |
| 12 |  |  |  | 13 | 0 |  | 0.8.11.13 | 0.1 | 0.6 | 0.10 | 0.13 | 0 | F |
| 13 |  |  |  |  | 0 | F | 0.4.8.11.13 | 0.1 | 0.6 | 0.10 | 0.13 | 0 | F |

Transition table of NFA above without $\varepsilon$-transitions.

Transition table of DFA which is equivalent to previous NFA.

DFA in this case has less states than the equivalent NFA. Q: Does it hold for any automaton of this type? Proof?

## Text search using NFA simulation without transform to DFA

```
Input: NFA, text in array t,
```

```
SetOfStates S = eps_CLOSURE(q0), S_tmp;
int i = 1;
while ((i <= t.length) && (!S.empty())) {
    for (q in S) // for each state in S
        if (q.isFinal)
        print(q.final_state_info); // pattern found
    S_tmp = Set.empty();
    for (q in S)
        S_tmp.union(eps_CLOSURE (delta(q, t[i]);));
    S = S_tmp;
    i++;
}
return S.containsFinalState(); // true or false
```

