

#### Languages, grammars, automata

#### Czech instant sources:

[1] Prof. Marie Demlová: A4B01JAG

http://math.feld.cvut.cz/demlova/teaching/jag/predn\_jag.html
Pages 1-27, in PAL, you may wish to skip: Proofs, chapters 2.4, 2.6, 2.8.

[2] I. Černá, M. Křetínský, A. Kučera: **Automaty a formální jazyky I** http://is.muni.cz/do/1499/el/estud/fi/js06/ib005/Formalni\_jazyky\_a\_automaty\_l.pdf Chapters 1 and 2, skip same parts as in [1].

#### English sources:

[3] B. Melichar, J. Holub, T. Polcar: **Text Search Algorithms**<a href="http://cw.felk.cvut.cz/lib/exe/fetch.php/courses/a4m33pal/melichar-tsa-lectures-1.pdf">http://cw.felk.cvut.cz/lib/exe/fetch.php/courses/a4m33pal/melichar-tsa-lectures-1.pdf</a>
Chapters 1.4 and 1.5, it is probably too short, there is nothing to skip.

[4] J. E. Hopcroft, R. Motwani, J. D. Ullman: Introduction to Automata Theory follow the link at http://cw.felk.cvut.cz/doku.php/courses/a4m33pal/literatura\_odkazy Chapters 1., 2., 3., there is a lot to skip, consult the teacher preferably.

For more references see PAL links page

http://cw.felk.cvut.cz/doku.php/courses/a4m33pal/literatura odkazy

Deterministic Finite Automaton (DFA)
Nondeterministic Finite Automaton (NFA)

Both DFA nd NFA consist of:

Finite input alphabet  $\Sigma$ .

Finite set of internal states Q.

One starting state  $q_0 \in Q$ .

Nonempty set of accept states  $F \subseteq Q$ .

Transition function  $\delta$ .

DFA transition function is  $\delta: Q \times \Sigma \rightarrow Q$ .

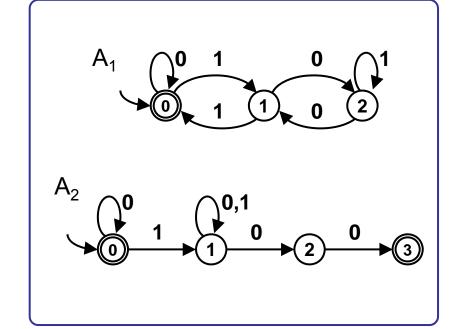
DFA is always in one of its states.

DFA transits from current state to another state depending on the current input symbol.

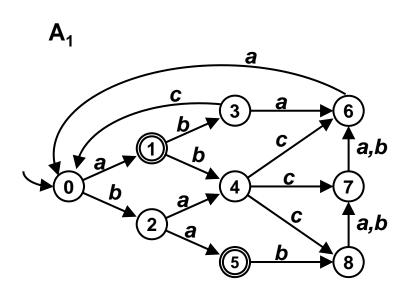
NFA transition function is  $\delta: \mathbb{Q} \times \Sigma \to P(\mathbb{Q})$  ( $P(\mathbb{Q})$  is powerset of  $\mathbb{Q}$ , set of all subsets of  $\mathbb{Q}$ )

NFA is always (simultaneously) in a set of any number of its states.

NFA transits from a state to a set of states depending on the current input symbol.

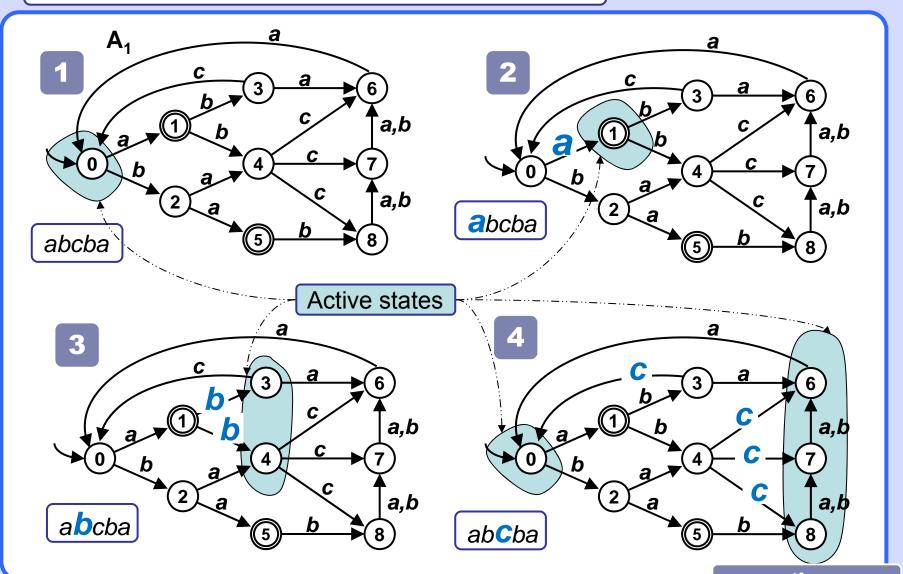


NFA  $A_1$ , its transition diagram and its transition table

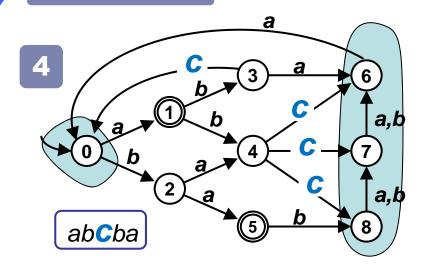


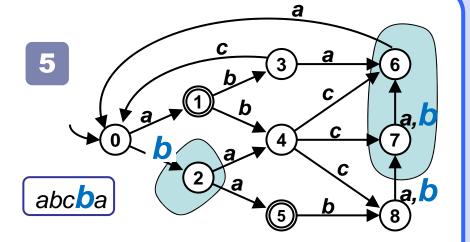
	а	b	C	_
0	1	2		
1		3,4		F
2	4,5 6			
3	6		0	
4			6,7,8	
2 3 4 5 6		8		F
	0			
7	6	6		
8	7	7		

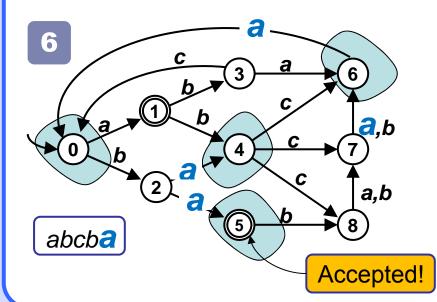
NFA A<sub>1</sub> processing input word *abcba* 



### ...continued







NFA A<sub>1</sub> has processed word *abcba* and went through read symbols and respective sets(!) of states

$$\{0\} \rightarrow a \rightarrow \{1\} \rightarrow b \rightarrow \{3, 4\} \rightarrow c \rightarrow \{0, 6, 7, 8\} \rightarrow b \rightarrow \{2, 6, 7\} \rightarrow a \rightarrow \{0, 4, 5, 6\}.$$

#### NFA simulation without transform to DFA

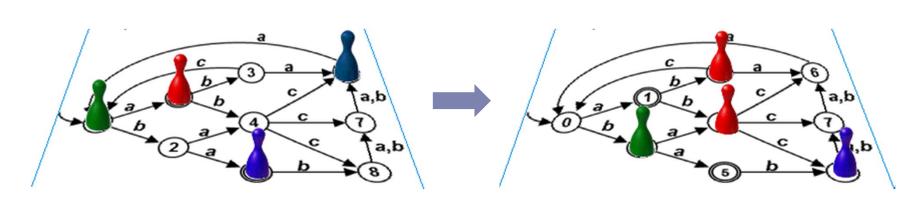
Each of current states is occupied by one token.

Read an input symbol and move tokens accordingly.

If token has more possibilities it will split into two or more tokens,

if token has no possibility it will leave the board, uhm, the transition diagram.

### Read **b** from input



#### NFA simulation without transform to DFA

Idea:

Register all states to which you have just arrived. In the next step read the input symbol x and move SIMULTANEOUSLY to ALL states to which you can get from ALL current states along transitions marked by x.

NFA to DFA

Algorithm

### Generating DFA A<sub>2</sub> equivalent to NFA A<sub>1</sub> using transition tables

```
Data
Each state of DFA is a subset of states of NFA
Start state of DFA is a one element set containing just start state of NFA.
A state of DFA is accept state iff it contains at least one accept state of NFA.
Construction
Create start state of DFA and corresponding first line of its transition table (TT).
For each state Q of DFA not yet processed do {
  Decompose Q into its constituent states Q1, ..., Qk of NFA
  For each symbol x of alphabet do {
   S = union of all references in NFA table at positions [Q1] [x], ... [Qk][x]
   if (S is not among states of DFA yet)
       add S to states of DFA and add corresponding line to TT of DFA
 Mark Q as processed
// Remember, empty set is also a set ot states, it can be easily a state of DFA
```

## **NFA to DFA**

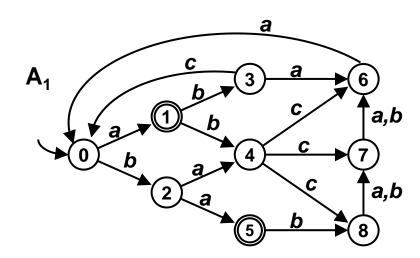
## Generating DFA $A_2$ equivalent to NFA $A_1$

	а	b	C	
0	1	2		]
1		3,4		F
2	4,5			
2 3 4 5 6 7	4,5 6		0	
4			6,7,8	
5		8		F
6	0			
7	6	6		]
8	7	7		

 $A_2$ 

### **Copy start state**

	а	b	С
0	1	2	



## NFA to DFA

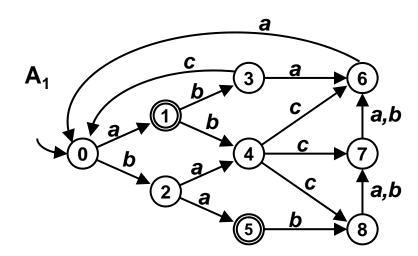
## Generating DFA $A_2$ equivalent to NFA $A_1$

	a	b	C	
0	1	2		
1		3,4		F
2	4,5 6			
2 3 4 5 6	6		0	
4			6,7,8	
5		8		F
6	0			
7	6	6		
8	7	7		]

 $A_2$ 

	a	b	C
0	1	2	
1		34	I

Add new state(s)



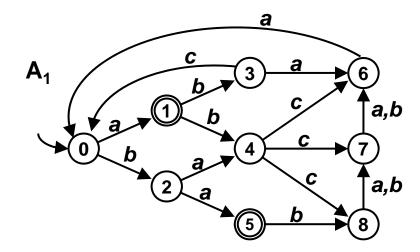
## Generating DFA $A_2$ equivalent to NFA $A_1$

	а	b	C	
0	1	2		]
1		3,4		F
	4,5			
2 3 4 5 6	4,5 6		0	]
4			6,7,8	]
5		8		F
6	0			]
7	6	6		]
8	7	7		

 $A_2$ 

	а	b	С
0	1	2	
1		34	
2	45		

Add new state(s)



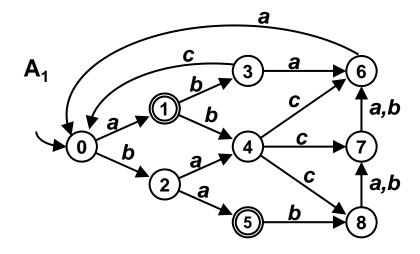
## Generating DFA A<sub>2</sub> equivalent to NFA A<sub>1</sub>

	а	b	C	_
0	1	2		
1		3,4		F
2	4,5			
3	6		0	
2 3 4 5 6			6,7,8	
5		8		F
6	0			
7	6	6		
8	7	7		]

 $A_2$ 

## Add new state(s)

	а	D	C
0	1	2	
1		34	
2	45		
34	6		0678
•••			



#### Note:

In the example we add the empty set to the table at the very end of the process just to keep the table uncluttered..

## Generating DFA A<sub>2</sub> equivalent to NFA A<sub>1</sub>

	а	b	C	_
0	1	2		
1		3,4		F
2	4,5			
3 4 5 6	6		0	
4			6,7,8	
5		8		F
6	0			
7	6	6		
8	7	7		

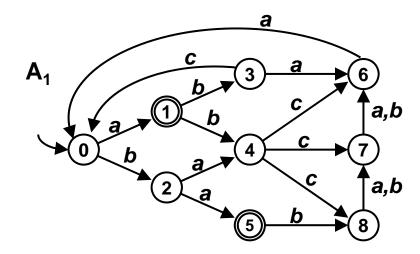
 $A_2$ 

34

45

## Add new state(s)

а	b	С	_
1	2		
	34		F
45			
6		0678	
	8	678	F



## Generating DFA A<sub>2</sub> equivalent to NFA A<sub>1</sub>

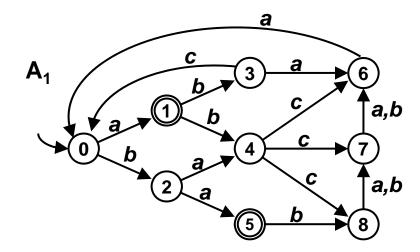
### ... after few more iterations...

Add new state(s)

	а	b	C	
0	1	2		
1		3,4		F
2	4,5			
2 3 4 5 6	6		0	
4			6,7,8	
5		8		F
6	0			
7	6	6		
8	7	7		1

A <sub>2</sub>

а	b	С	
1	2		1
	34		F
45			
6		0678	
	8	678	F
0			
0167	267		
7	7		
067	67		
016	2346		F
0456	6		
6	6		1



С

n

0678

678

n

n

0678

678

678

n

n

n

n

678

n

F

## **NFA to DFA**

a

1

4,5

6

0

6

7

0

2

3

4

5

6

8

## DFA A<sub>2</sub> equivalent to NFA A<sub>1</sub>

b

2

3,4

8

6

7

C

0

6,7,8

# $\mathbf{A_2}$

...FINISHED!

0

a

2

34

45

1	2	n
n	34	n
45	n	n
6	n	0678
n	8	678
0	n	n
0167	267	n
7	7	n
067	67	n
016	2346	n
0456	6	n
6	6	n
016	2346	n
06	6	n

234

n

28

2

234

n 7

8

68 7

26

34

n

28

n

b

	ľ

457	
0	
6	
07	
16	
0	
045	
	_

n

01

0456

01

01

456

45
6
0678
8
678
0167
267
7
067
67
016
2346
0456
06
01
234
28
456
457
68
07
16
_
26
654
n

	a	
$A_1$	c 3	<b>a</b> 6
		c a,b
0 6	a <b>4</b>	<b>c</b> → (7)
	2 a	c a,b
	5	<b>b</b> 8

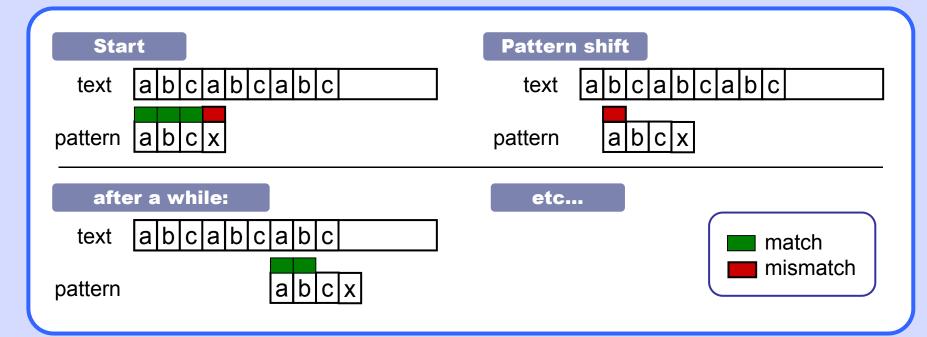
**Text Search** 

## Naïve approach

#### To be used with great caution!

- 1. Align pattern with the beginning of text.
- 2. While corresponding symbols of pattern and text match each other move forward by one symbol in pattern.
- 3. When symbol mismatch occurs shift pattern forward by one symbol, reset position in the pattern to the beginning of pattern and go to 2.
- 4. When the end of pattern is passed report success, reset position in the pattern to its beginning and go to 2.
- 5. When the end of text is reached stop.

Might be both efficient and not



### **Text Search**

**Alphabet:** Finite set of symbols.

**Text:** Sequence of symbols of the alphabet.

**Pattern**: Sequence of symbols of the same alphabet,

pattern occurence is to be detected in the text

Text is often fixed or seldom changed, pattern typically varies (looking for different words in the same document), pattern is often significantly shorter than the text.

#### **Notation**

Alphabet:  $\Sigma$ 

Symbols in the text:  $t_1, t_2, \dots t_n$ 

Symbols in the pattern:  $p_1, p_2, \dots p_m$ 

Holds  $m \le n$ , usually m << n

#### **Example**

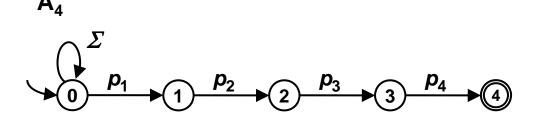
Text: ...task is very simple but it is used very freq...

Pattern: simple

NFA A<sub>3</sub> which accepts just a single word  $p_1p_2p_3p_4$ .

$$A_3 \longrightarrow 0 \xrightarrow{p_1} 1 \xrightarrow{p_2} 2 \xrightarrow{p_3} 3 \xrightarrow{p_4} 4$$

NFA  $A_4$  which accepts each word with suffix  $p_1$   $p_2$   $p_3$   $p_4$  with its transition table.

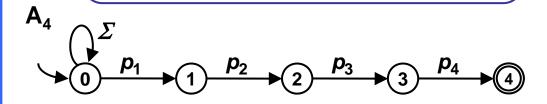


	$p_1$	$p_2$	$p_3$	$p_4$	Z
0	0,1	0	0	0	0
1		2			
2			3		
3				4	
4					

 $z \in \Sigma - \{p1, p2, p3, p4\}$ 

#### repeated

NFA  $A_4$  which accepts each word with suffix  $p_1$   $p_2$   $p_3$   $p_4$  and its transition table.

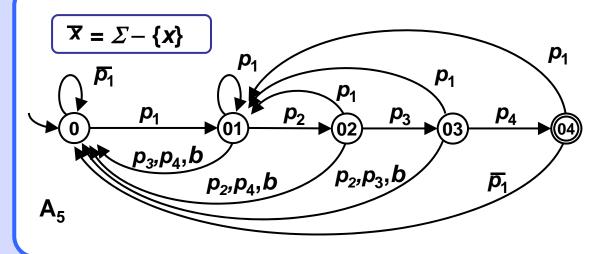


	$p_1$	$p_2$	$p_3$	$p_4$	Z
0	0,1	0	0	0	0
1		2			
2			3		
3				4	
4					

 $z \in \Sigma - \{p1, p2, p3, p4\}$ 

#### equivalently

DFA  $A_5$  is a deterministic equivalent of NFA  $A_4$ .

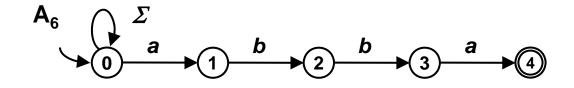


	$p_1$	$p_2$	$p_3$	$p_4$	Z
0	01	0	0	0	0
01	01	02	0	0	0
02	01	0	03	0	0
03	01	0	0	04	0
04	01	0	0	0	0

-

#### example

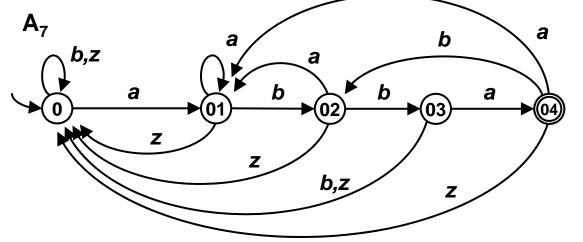
NFA A<sub>6</sub> which accepts each word with suffix *abba* with its transition table



	а	b	Z	
0	0,1	0	0	
1		2		
2		3		
2 3 4	4			
4				F

$$z \in \Sigma - \{a, b\}$$

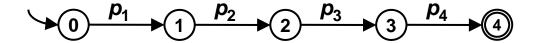
DFA  $A_7$  is a deterministic equivalent of NFA  $A_6$ . It also accepts each word with suffix *abba*.



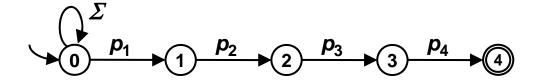
	a	b	Z
0	01	0	0
01	01	02	0
02	01	03	0
03	014	0	0
04	01	02	0

F

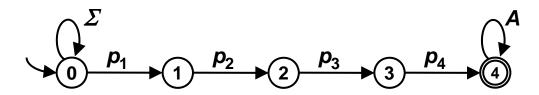
NFA accepting exactly one word  $p_1p_2p_3p_4$ .



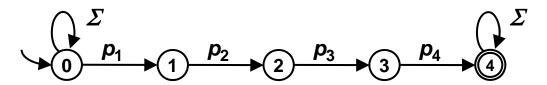
NFA accepting any word with suffix  $p_1p_2p_3p_4$ .



NFA accepting any word with substring (factor)  $p_1p_2p_3p_4$  anywhere in it.

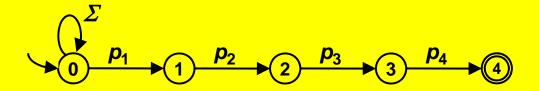


NFA accepting any word with substring (factor)  $p_1p_2p_3p_4$  anywhere in it.



Can be used for search, but the following reduction is usual.

Text search NFA for finding pattern  $P = p_1p_2p_3p_4$  in the text.



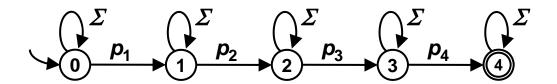
NFA stops when pattern is found.

Want to know the position of the pattern in the text? Equip the transitions with a counter.

$$[pos=0] \xrightarrow{\Sigma, [pos++]} \underbrace{p_1 \qquad p_2 \qquad p_3 \qquad p_4}_{0}$$

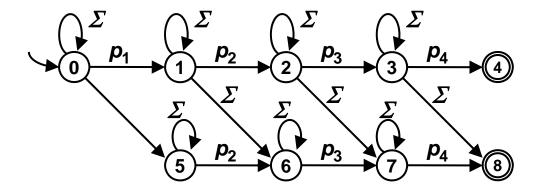
### **Example**

NFA accepting any word with subsequence  $p_1p_2p_3p_4$  anywhere in it.



### **Example**

NFA accepting any word with subsequence  $p_1p_2p_3p_4$  anywhere in it, one symbol in the sequence may be altered.



Alternatively: NFA accepting any word containing a subsequence Q which Hamming distance from  $p_1p_2p_3p_4$  is at most 1.

## **Languages Hierarchy**

Search NFA can search for more than one pattern simultaneously. The number of patterns can be **finite** -- this leads to dictionary automaton (we will meet them later) **or infinite** -- this leads to regular language.

### **Chomsky language hierarchy remainder**

Grammar	Language	Automaton
Type-0 Type-1	Recursively enumerable Context-sensitive	Turing machine Linear-bounded
Type-2 Type-3	Context-free Regular	non-deterministic Turing machine Non-deterministic pushdown automaton Finite state automaton (NFA or DFA)

Only regular languages can be processed by NFA/DFA. More complex languages cannot. For example any language containing *well-formed parentheses* is context-free and not regular and cannot be recognized by NFA/DFA.

#### **Operations on regular languages**

Let  $L_1$  and  $L_2$  be any languages. Then  $L_1 \cup L_2$  is union of  $L_1$  and  $L_2$ . It is a set of all words which are in  $L_1$  or  $L_2$ .  $L_1.L_2$  is concatenation of  $L_1$  and  $L_2$ . It is a set of all words w for which holds  $w = w_1w_2$  (concatenation of words  $w_1$  and  $w_2$ ), where  $w_1 \in L_1$  and  $w_2 \in L_2$ .  $L_1^*$  is Kleene star or Kleene closure of language  $L_1$ . It is set of all words which are concatenations of any number (incl. zero) of any words of  $L_1$  in any order.

#### Closure

Whenever  $L_1$  and  $L_2$  are regular languages then  $L_1 \cup L_2$ ,  $L_1.L_2$ ,  $L_1^*$  are regular languages too.

### **Example**

```
\begin{array}{l} L_1 = \{001,\,0001,\,00001,\,...\},\,L_2 = \{110,\,1110,\,11110,\,...\},\\ L_1 \cup L_2 = \{001,\,110,\,0001,\,1110,\,0001,\,1110,\,...\}\\ L_1.L_2 = \{001110,\,00111110,\,00111110,\,...,\,0001110,\,00011110,\,000111110,\,...\,\}\\ L_1^* = \{\mathcal{E},\,001,\,001001,\,001001001,\,...\,\,0010001,\,00100010001,\,...\\ ...,\,00100001,\,001000001,\,...\,\}\,//\,\, this \,\, one \,\, is \,\, not \,\, easy \,\, to \,\, list \,\, nicely \,\,...\,\, or \,\, is \,\, it? \end{array}
```

### Regular expressions defined recursively

Symbol  $\varepsilon$  is regular expression.

Each symbol of alphabet  $\Sigma$  is regular expression.

Whenever  $e_1$  and  $e_2$  are regular expressions also strings  $(e_1)$ ,  $e_1+e_2$ ,  $e_1e_2$ ,  $(e_1)^*$  are regular expressions.

### Languages represented by regular expressions (RE) defined recursively

RE  $\varepsilon$  represents language containing only empty string

RE x, where  $x \in \Sigma$ , represents language  $\{x\}$ .

Let RE's e<sub>1</sub> and e<sub>2</sub> represent languages L<sub>1</sub> and L<sub>2</sub>. Then

RE (e<sub>1</sub>) represents L<sub>1</sub>, RE e<sub>1</sub>+e<sub>2</sub> represents L<sub>1</sub>  $\cup$  L<sub>2</sub>, RE e<sub>1</sub>e<sub>2,</sub> represents L<sub>1</sub>.L<sub>2</sub>, RE (e<sub>1</sub>)\* represents L<sub>1</sub>\*.

### **Examples**

```
0+1(0+1)^* all integers in binary without leading 0's 0.(0+1)^*1 all finite binary fractions \in (0, 1) without trailing 0's ((0+1)(0+1+2+3+4+5+6+7+8+9) + 2(0+1+2+3)):(0+1+2+3+4+5)(0+1+2+3+4+5+6+7+8+9) all 1440 day's times in format hh:mm from 00:00 to 23:59 (mon+(wedne+t(ue+hur))s+fri+s(atur+un))day English names of days in the week (1+2+3+4+5+6+7+8+9)(0+1+2+3+4+5+6+7+8+9)^*((2+7)5+(5+0)0) all decimal integers \geq 100 divisible by 25
```

## **Regular Expressions**

#### **Convert regular expression to NFA**

Input: Regular expression R containing *n* characters of the given alphabet. Output: NFA recognizing language L(R) described by R.

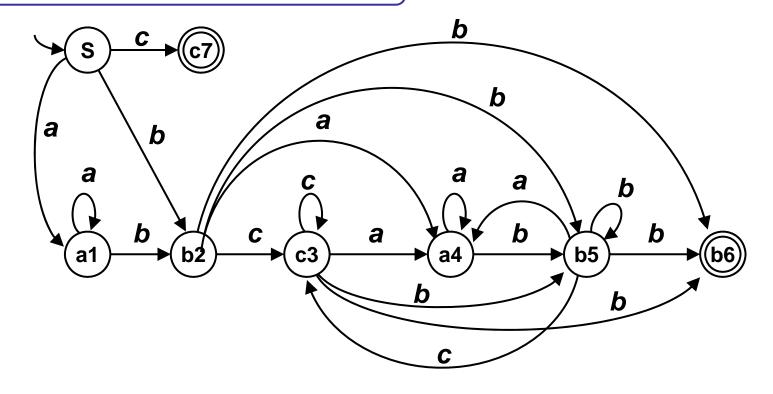
```
Create start state S
for each k (1 \le k \le n) {
  assign index k to the k-th character in R
   // this makes all characters in R unique: c[1], c[2], ..., c[n].
  create state S[k] // S[k] corresponds directly to c[k]
for each k (1 \le k \le n) {
  if c[k] can be the first character in some string described by R
    then create transition S \to S[k] labeled by c[k] with index stripped off
  if c[k] can be the last character in some string described by R
    then mark S[k] as final state
  for each p (1 \le p \le n)
    <u>if</u> (c[k] can follow immediately after c[p] in some string described by R)
      then create transition S[p] \rightarrow S[k] labeled by c[k] with index stripped off
```

## Regular expression

R = a\*b(c + a\*b)\*b + cAdd indices:

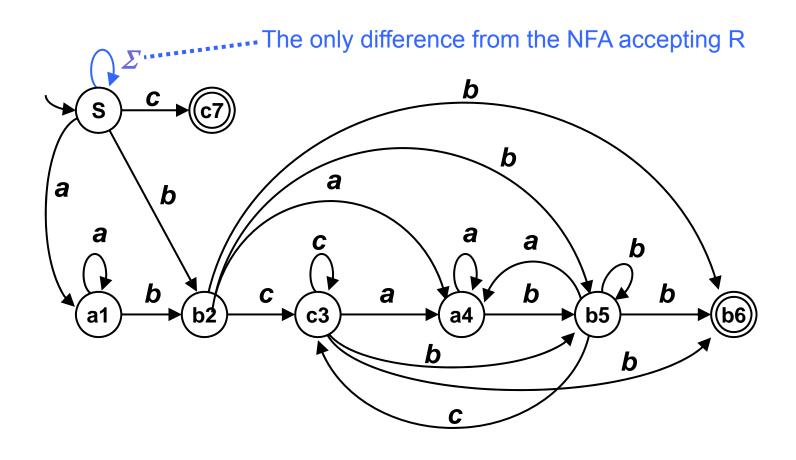
$$R = a_1^* b_2(c_3 + a_4^* b_5)^* b_6 + c_7$$

## NFA accepts L(R)



NFA searches the text for any occurrence of any word of L(R)

$$R = a*b (c + a*b)*b + c$$



#### **Bonus**

To find a subsequence representing a word  $\in$  L(R), where R is a regular expression, do the following:

Create NFA acepting L(R)

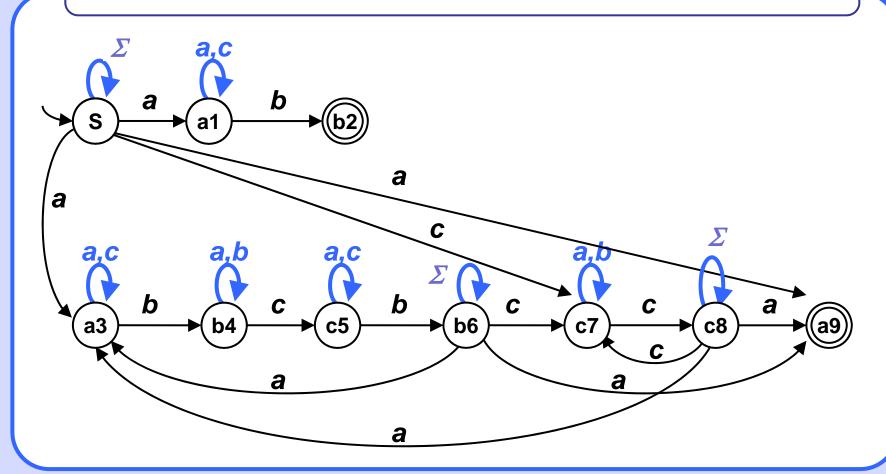
Add self loops to the states of NFA:

- 1. Self loop labeled by  $\Sigma$  (whole alphabet) at the start state.
- 2. Self loop labeled  $\Sigma \{x\}$  at each state which outgoing transition(s) are labeled by single  $x \in \Sigma$ . // serves as an "optimized" wait loop
- 3. Self loop labeled by  $\Sigma$  at each state which outgoing transition(s) are labeled by more than single symbol from  $\Sigma$ . // serves as an "usual" wait loop
- 4. No self loop to all other states. // which have no outgoing loop, final ones

#### **Bonus**

NFA searches the text for any occurrence of any subsequence representing a word word of L(R)

$$R = ab + (abcb + cc)^* a$$



Transforming NFA which searches text for an occurrence of a word of a given regular language into the equivalent DFA might take exponential space and thus also exponential time. Not always, but sometimes yes:

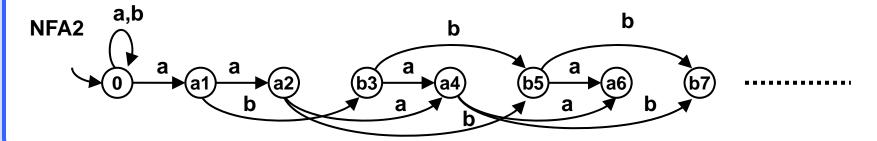
Consider regular expression R = a(a+b)(a+b)...(a+b) over alphabet  $\{a, b\}$ .

#### **Text search NFA1 for R**

NFA1 a,b a

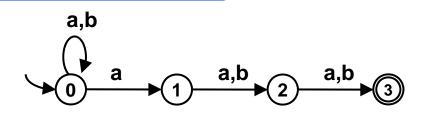
#### **Mystery**

Text search NFA2 for R, why not this one?



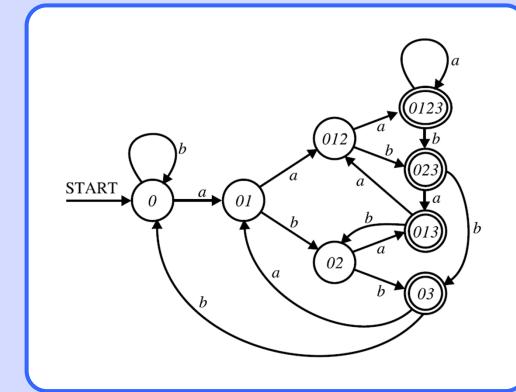
$$R = a(a+b)(a+b)$$

### **Text search NFA for R**



### **NFA** table

	a	b
0	0,1	0
1	2	2
2	3	3
3	_	-



### **DFA** table

	a	b
0	01	0
01	012	02
012	0123	023
0123	0123	023
02	013	03
023	013	03
013	012	02
03	01	0

### Search the text for more than just exact match

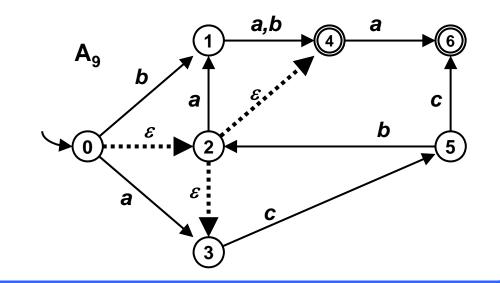
#### NFA with $\varepsilon$ -transitions

The transition from one state to another can be performed **without** reading any input symbol. Such transition is labeled by symbol  $\varepsilon$ .

#### *⊱*-closure

Symbol  $\varepsilon$ –CLOSURE(p) denotes the set of all states q, which can be reached from p using only  $\varepsilon$ –transitions. By definition let  $\varepsilon$ –CLOSURE(p) = {p}, when there is no  $\varepsilon$ –transition out from p.

 $\varepsilon$ -CLOSURE(0) = {2, 3, 4}  $\varepsilon$ -CLOSURE(1) = {1}  $\varepsilon$ -CLOSURE(2) = {3, 4}  $\varepsilon$ -CLOSURE(3) = {3} ...

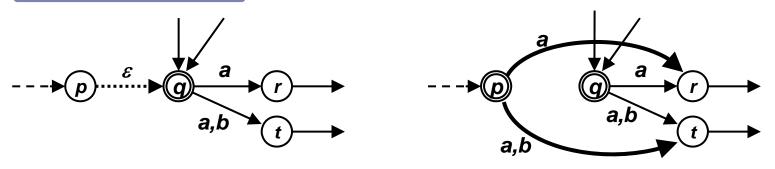


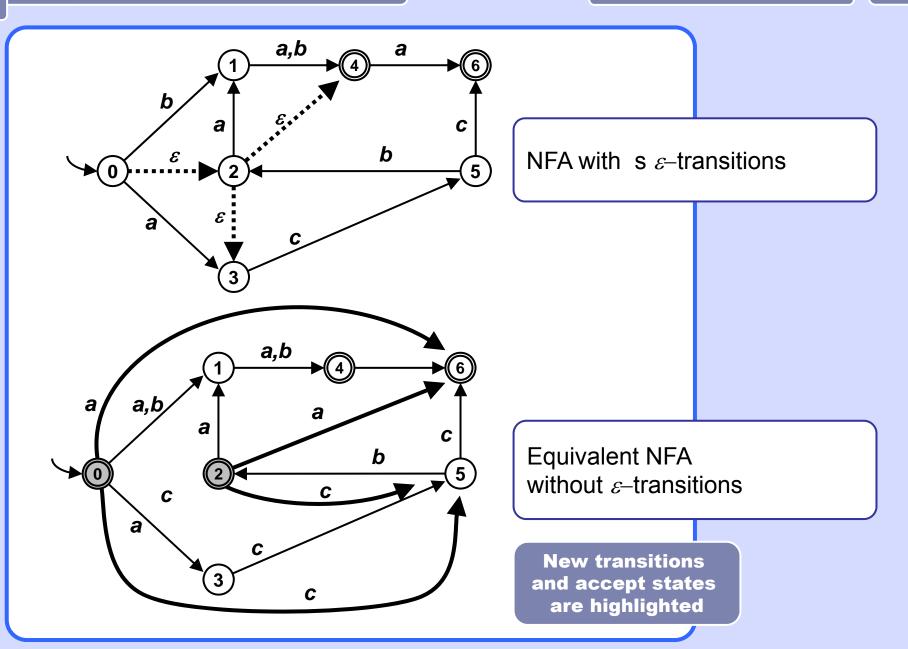
### Construction of equivalent NFA without $\varepsilon$ -transitions

Input: NFA A with some  $\varepsilon$ -transitions. Output: NFA A' without  $\varepsilon$ -transitions.

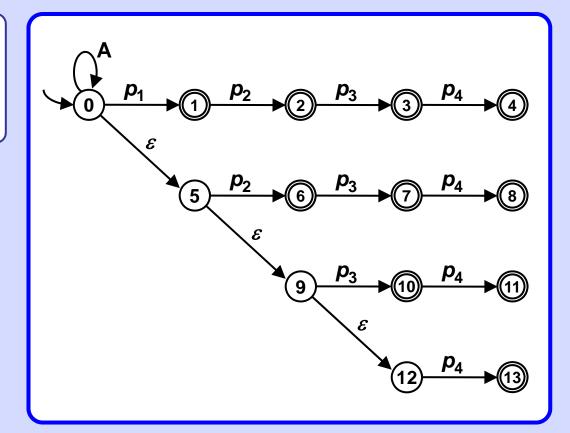
- 1. A' = exact copy of A.
- 2. Remove all  $\varepsilon$ -transitions from A'.
- 3. In A' for each (q, a) do: add to the set  $\delta(p,a)$  all such states r for which holds  $q \in \varepsilon$ -CLOSURE(p) and  $\delta(q,a) = r$ .
- 4. Add to the set of final states F in A' all states p for which holds  $\varepsilon$ -CLOSURE(p)  $\cap$   $F \neq \emptyset$ .

#### easy construction





NFA for search for any unempty substring of pattern  $p_1p_2p_3p_4$  over alphabet  $\Sigma$ . Note the  $\varepsilon$ -transitions.



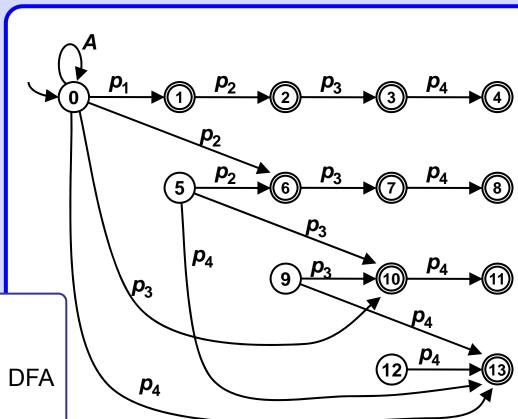
### Powerful trick!

**Union** of two or more NFA:

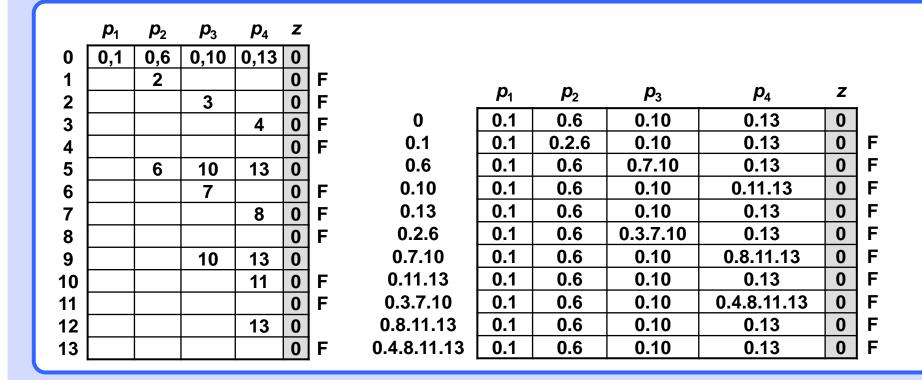
Create additional start state S and add  $\varepsilon$ —transitions from S to start states of all involved NFA's. Draw an example yourself!

## **Epsilon Transitions**

Equivalent NFA for search for any unempty substring of pattern  $p_1p_2p_3p_4$  with  $\varepsilon$ -transitions removed.



States 5, 9, 12 are unreachable. Transformation algorithm NFA -> DFA if applied, will neglect them.



Transition table of NFA above without  $\varepsilon$ -transitions.

Transition table of DFA which is equivalent to previous NFA.

DFA in this case has less states than the equivalent NFA.

Q: Does it hold for any automaton of this type? Proof?

### Text search using NFA simulation without transform to DFA

Input: NFA, text in array t,

```
SetOfStates S = eps CLOSURE(q0), S tmp;
int i = 1;
while ((i <= t.length) && (!S.empty())) {
 for (q in S) // for each state in S
    if (q.isFinal)
     print(q.final_state_info); // pattern found
 S tmp = Set.empty();
 for (q in S)
    S_tmp.union(eps_CLOSURE(delta(q, t[i])));
 S = S_{tmp}
  i++;
return S.containsFinalState(); // true or false
```