Advanced algorithms

Seminar 9/19/2012

All the following problems can be solved by a 'brute-force' method - just by trying it for low values of N - and 'noticing' a pattern. The proof that the pattern holds for all values of N is the hard part, and just by trying, you may learn a new stuff.

1. Light bulbs

This could be solved by simulating the pressing of switches and writing down the state of the corresponding light bulbs. What one may notice is that light bulbs which remained turned on are on the position of a square number, i.e. the 1-st, 4-th, 9-th, ..., $(\lfloor\sqrt{N}\rfloor)^2$. Therefore, the answer to the question of how many light bulbs remained turned on is $\lfloor\sqrt{N}\rfloor$.

To proof it just consider how many times a bulb needs to be toggled to remain turned on – the answer is odd number of times (why?). To finish the proof, we need to show that every non-square number has even number of divisors and square numbers have odd number of divisors – each divisor corresponds to the ordinal number of a switch that operates the light bulb.

The excellent thing about divisors of a number is that they pair up such that the product of the pair is the number itself. For example consider the number 18 whose list of divisors is $1, 2, 3, 6, 9, 18$, hence

$$\begin{array}{ll} 18 = 1 \times 18 & (18 = 18 \times 1) \\ 18 = 2 \times 9 & (18 = 9 \times 2) \\ 18 = 3 \times 6 & (18 = 6 \times 3) \end{array}$$

As you can see, the 18-th light bulb would remain turned off. Let us take a square number this time, such as 36 - list of divisors is $1, 2, 3, 4, 6, 9, 12, 18, 36$, so:

$$\begin{array}{ll} 36 = 1 \times 36 & (36 = 36 \times 1) \\ 36 = 2 \times 18 & (36 = 18 \times 2) \\ 36 = 3 \times 12 & (36 = 12 \times 3) \\ 36 = 4 \times 9 & (36 = 9 \times 4) \\ 36 = 6 \times 6 & \end{array}$$

This time the 36-th light bulb would remain turned on. The reason is obvious from the table above – and it holds for any square number K^2, the pair represented by roots, i.e. (K, K), is counted only once, hence, the number of divisors of K^2 is odd.

2. Chocolate

This is a relatively easy problem, whose solution lies in the answer to the following question: What happens after you break a piece? When you take a piece and break it, you end up with 1 more piece then you had before the breaking – one represents the former piece, the other the chipped off piece. Therefore, after performing K breaks, you end up with $K + 1$ pieces. Now solve for K to get the answer.

Someone came up with an idea to put the pieces on top of each other. In this case the breaking can be done much faster. Can you find you what would be the number of breaks in this case?

3. Garden

One of those really nice problems whose answer is a number from the Fibonacci sequence. How come? Let us say that the number of ways to plant the garden with N patches is $T(N)$. Then the resulting gardens can be divided into two disjoint(!) groups:

(1) Gardens which start with a patch containing carrots.

(2) Gardens which start with a patch containing cabbages.

The number of gardens in the case (1) is $T(N-1)$ since it remains to plant on $N-1$ patches. But the number of gardens in the case (2) is not $T(N-1)$ because the 2nd patch cannot contain cabbages (it would look ugly then!), therefore, it will contain carrots and the number of the gardens drop down to $T(N-2)$.

This brings us to the conclusion that $T(N) = T(N-1) + T(N-2)$. Now it just remains to find out the base cases for $N = 1$ and $N = 2$. Hurray!

4. Tutor

You may find the answer in here:

`http://en.wikipedia.org/wiki/Coupon_collector's_problem`