photograph itself (in contrast to the algorithm presented here), but is typically between 6 and 10 times. These are compression ratios that are comparable to those we have just calculated. Compression using iterated function systems has been studied for quite some time but is not used in practice. Its weak point is the amount of time required to compress an image. (Recall that in our earliest discussion of the algorithm the number of steps was proportional to the square of the number of pixels, $(h \times v)^2$. In comparison, the complexity of the JPEG algorithm grows only linearly with image size, and is proportional to $h \times v$. For a photographer in the field snapping photos one after the other, this is a big advantage. For research images being processed on a high-powered computer, it is less so. Regardless, the domain moves quite fast, and iterated function systems may not have spoken their last words.

11.8 Exercises

Certain of the following fractals have been constructed based on the figures found in [1].

1. (a) For the fractals of Figure 11.12, find iterated function systems describing them. In each case clearly specify the coordinate system you have chosen. Afterward, reconstruct each of the figures in software.
(b) Given your chosen coordinate system, find two different iterated function systems describing the fractal (b).

2. For the fractals of Figure 11.13, find iterated function systems describing them. In each case clearly specify the coordinate system you have chosen. Afterward, reconstruct each of the figures in software.

3. For the fractals of Figure 11.14, find iterated function systems describing them. In each case clearly specify the coordinate system you have chosen. Afterward, reconstruct each of the figures in software. Attention: here the triangle in Figure 11.14(b) is equilateral, in contrast to the Sierpiński triangle in our earlier example.

4. For the fractals of Figure 11.15, find iterated function systems describing them. In each case clearly specify the coordinate system you have chosen. Afterward, reconstruct each of the figures in software.

5. Amuse yourself by constructing arbitrary iterated function systems and trying to intuit their attractors. Afterward, confirm or disprove your intuitions by plotting them on a computer.

6. Calculate the fractal dimensions of the fractals in Exercises 1 (except (a)), 2, 3, and 4. (In certain cases you will be required to pursue numeric approaches.)

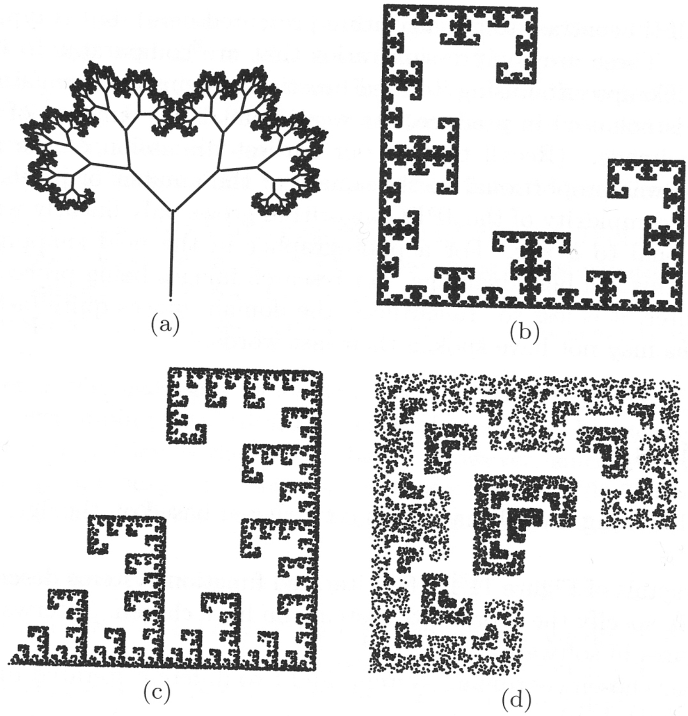


Fig. 11.12. Exercise 1.

7. The Cantor set is a subset of the unit interval $[0,1]$. It is obtained as the attractor of the iterated function system $\{T_1, T_2\}$, where T_1 and T_2 are the affine contractions defined by $T_1(x) = x/3$ and $T_2(x) = x/3 + 2/3$.
(a) Describe the Cantor set.
(b) Draw the Cantor set. (You may pursue the first few iterations by hand, but it is easiest to use a computer.)
(c) Show that there exists a bijection between the Cantor set and the set of real numbers with base-3 expansions of the form

$$0.a_1 a_2 \ldots a_n \ldots,$$

where $a_i \in \{0, 2\}$.
(d) Calculate the fractal dimension of the Cantor set.

8. Show that the fractal dimension of the Cartesian product $A_1 \times A_2$ is the sum of the fractal dimensions of A_1 and A_2:

$$D(A_1 \times A_2) = D(A_1) + D(A_2).$$

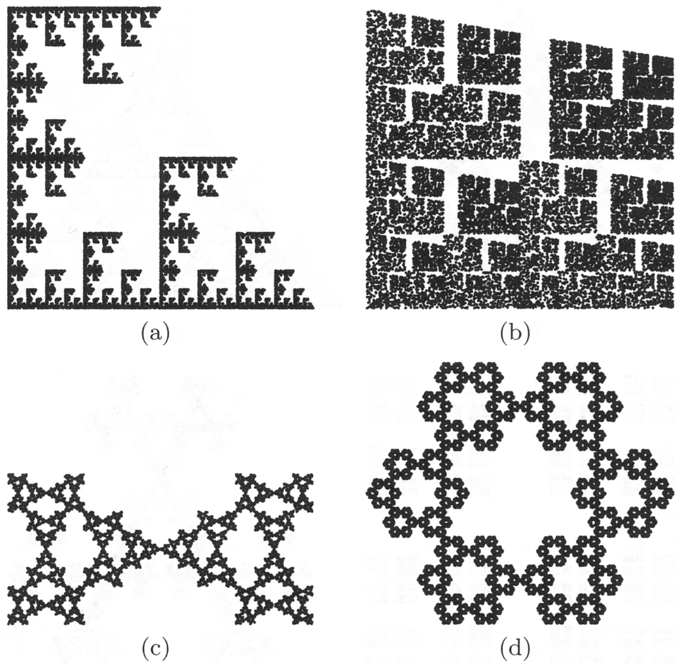


Fig. 11.13. Exercise 2.

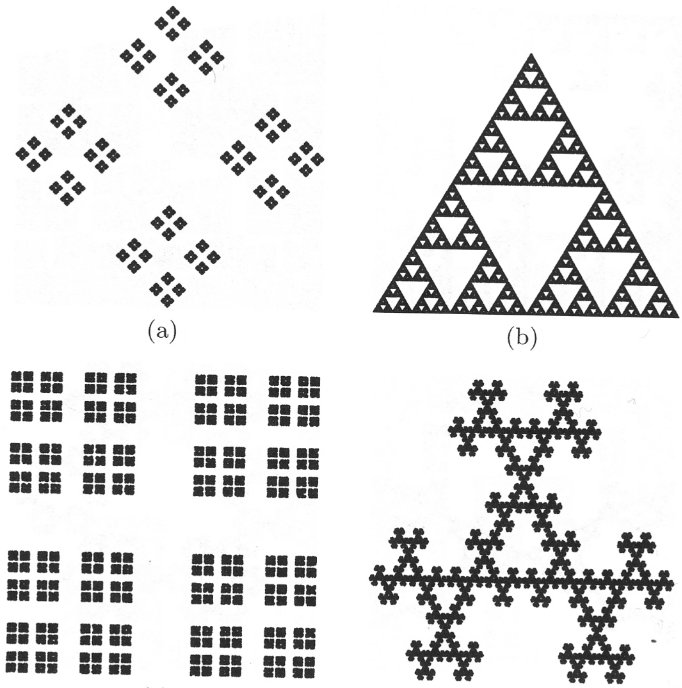
9. Let A be the Cantor set, as described in Exercise 7. This is a subset of $\mathbb{R}$. Find an iterated function system on $\mathbb{R}^2$ whose attractor is $A \times A$.

10. The Koch snowflake (or von Koch snowflake) is constructed as the limiting object of the following process (see Figure 11.16):

- Begin with the segment $[0, 1]$.
- Replace the initial segment with four segments, as shown in Figure 11.16(b)).
- Iterate the process, at each step replacing each segment by four smaller segments (see Figure 11.16(c)).

(a) Give an iterated function system that constructs the von Koch snowflake.
(b) Can you give an iterated function system for building the von Koch snowflake that requires just two affine contractions?
(c) Calculate the fractal dimension of the von Koch snowflake.

11. Explain how to modify an iterated function system on $\mathbb{R}^2$



(a) (b) (c) (d)

Fig. 11.14. Exercise 3.

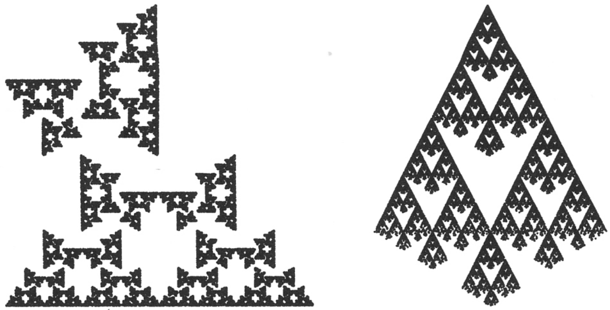
(a) such that its attractor will be twice as large in both dimensions;
(b) to translate the location of its bottom leftmost point.

12. Consider an affine transformation $T(x,y) = (ax+by+e, cx+dy+f)$.
(a) Show that T is an affine contraction if and only if the associated linear transformation $U(x,y) = (ax+by, cx+dy)$ is a contraction.
(b) Show that U contracts distances if

$$\begin{cases} a^2+c^2<1, \\ b^2+d^2<1, \\ a^2+b^2+c^2+d^2-(ad-bc)^2<1. \end{cases}$$

Suggestion: it suffices to show that the square of the length of $U(x,y)$ is less than the square of the length of (x,y) for all nonzero (x,y).

13. Let $P_1,\ldots,P_4$ be four noncoplanar points in $\mathbb{R}^3$. Let $Q_1,\ldots,Q_4$ be four other points of $\mathbb{R}^3$. Show that there exists a unique affine transformation $T:\mathbb{R}^3\to\mathbb{R}^3$ such that $T(P_i)=Q_i$.



(a) (b)

Fig. 11.15. Exercise 4.

(a) The initial segment (b) The first iteration (c) The second iteration (d) The von Koch snowflake

Fig. 11.16. Constructing the von Koch snowflake of Exercise 10.

Remark: We can consider systems of iterated functions in $\mathbb{R}^3$. As an example, we could use an iterated function system in this space to describe a fern leaf bent under its own weight. We could then project this image to the plane in order to display it.

14. Consider $v \in \mathbb{R}^2$ and A, B, two closed and bounded subsets of $\mathbb{R}^2$. Show that $d(v, A \cup B) \leq d(v, A)$ and $d(v, A \cap B) \geq d(v, A)$.

15. Proceeding numerically, find the contraction factors of the individual transforms T_i for the fern leaf. Are any of these exact contraction factors?

16. **(a)** Let B_1 and B_2 be two disks in $\mathbb{R}^2$ with radius r, and whose centers are at a distance of d from each other. Calculate $d_H(B_1, B_2)$.
(b) Let B_1 and B_2 be two concentric disks in the plane with radii r_1 and r_2, respectively. Calculate $d_H(B_1, B_2)$.