Data structures and algorithms

Part 9

Searching and Search Trees II

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Exploited in Advanced Algorithms 2012-2015

10.12. 2007

Topics

Red-Black tree
- Insert
- Delete

B-Tree
- Motivation
- Search
- Insert
- Delete

Based on:
[Cormen, Leiserson, Rivest: Introduction to Algorithms, Chapter 14 and 19, McGraw Hill, 1990]
[Whitney: CS660 Combinatorial Algorithms, San Diego State University, 1996]
[Frederic Maire: An Introduction to Btrees, Queensland University of Technology,1998]

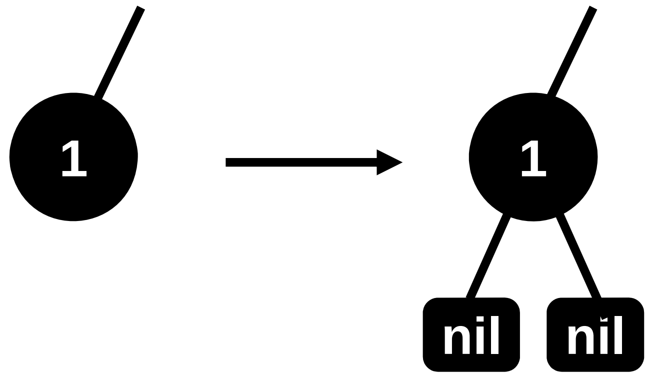
Red-Black tree

Approximately balanced BST

$h_{RB} \leq 2 \times h_{BST}$ (height ≤ 2 x height of a balanced tree)

Additional bit for COLOR = {red | black}

nil (non-existent child) = pointer to **nil** node



leaf → inner node

Red-Black tree

A binary search tree is a red-black tree if:

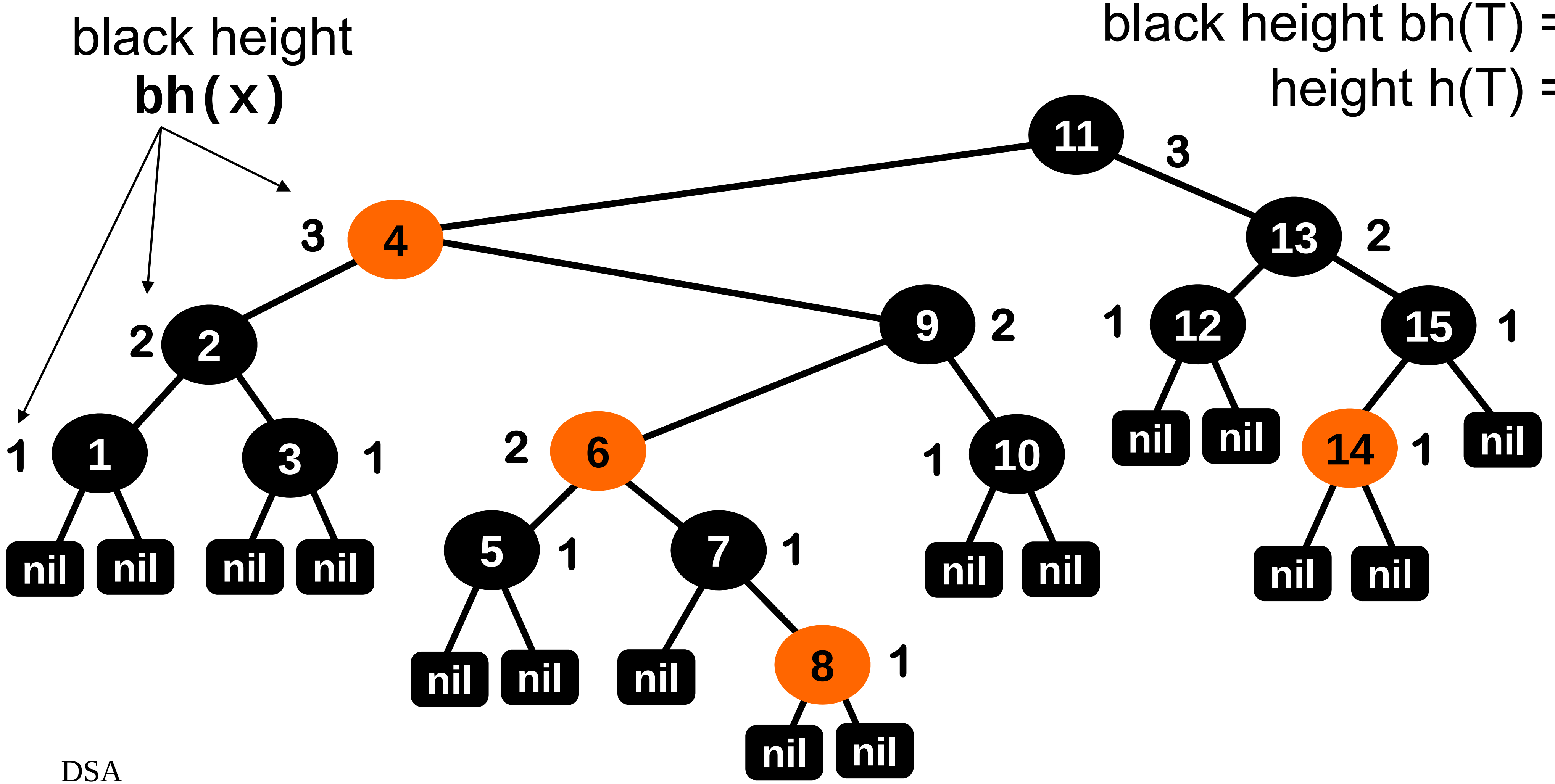
1. Every node is either red or black.
2. Every leaf (nil) is black.
3. If a node is red, then both its children are black.
4. Every simple path from a node to a descendant leaf contains the same number of black nodes.
5. Root is black.

Black-height **bh(x)** of a node **x** is the number of black nodes on any path from **x** to a leaf, not counting **x**

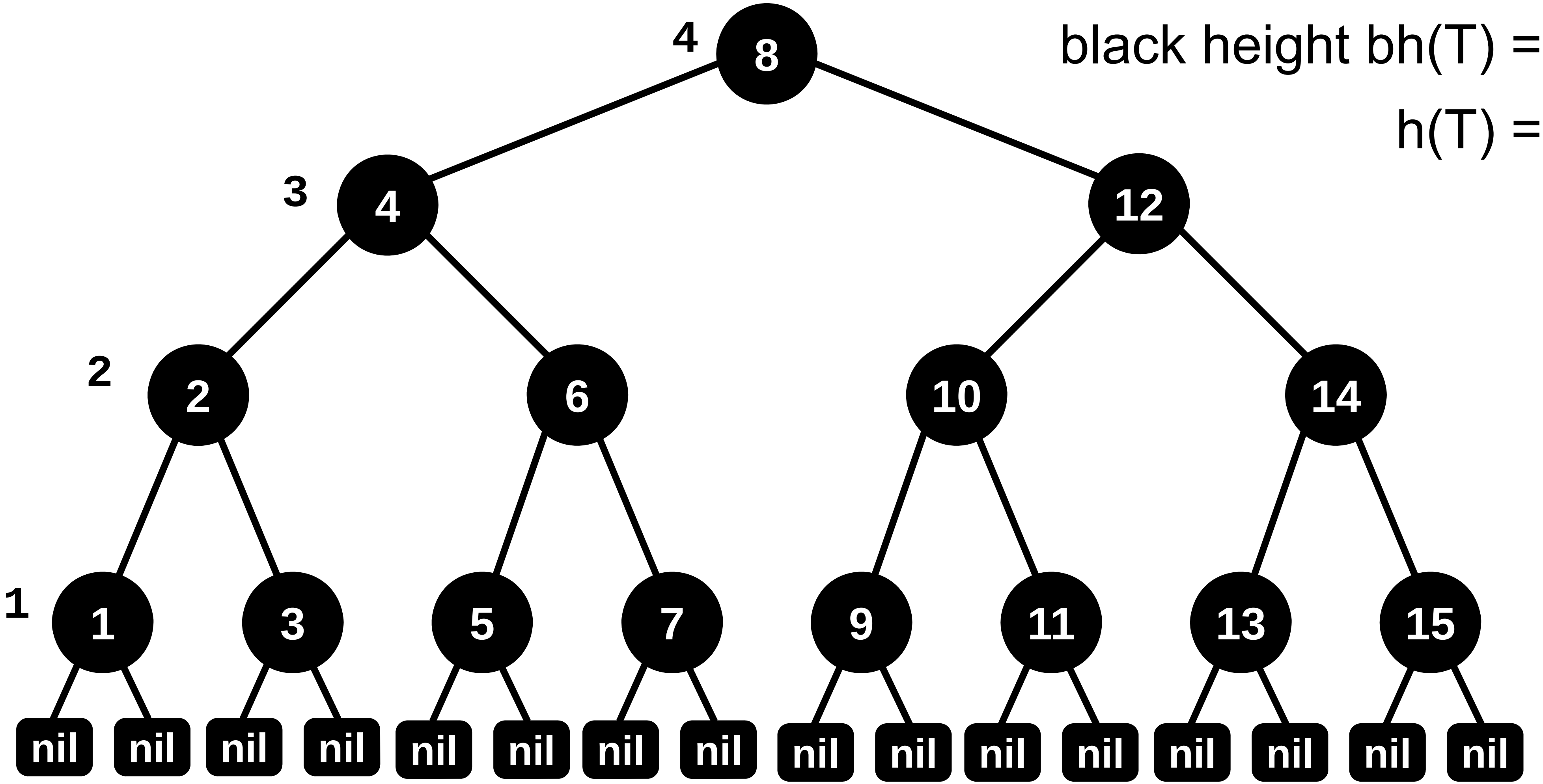
Red-Black tree

Black-height bh(x) of a node **x** is the number of black nodes on any path from **x** to a leaf, not counting **x**.

black height bh(T) = 3
height h(T) = 6



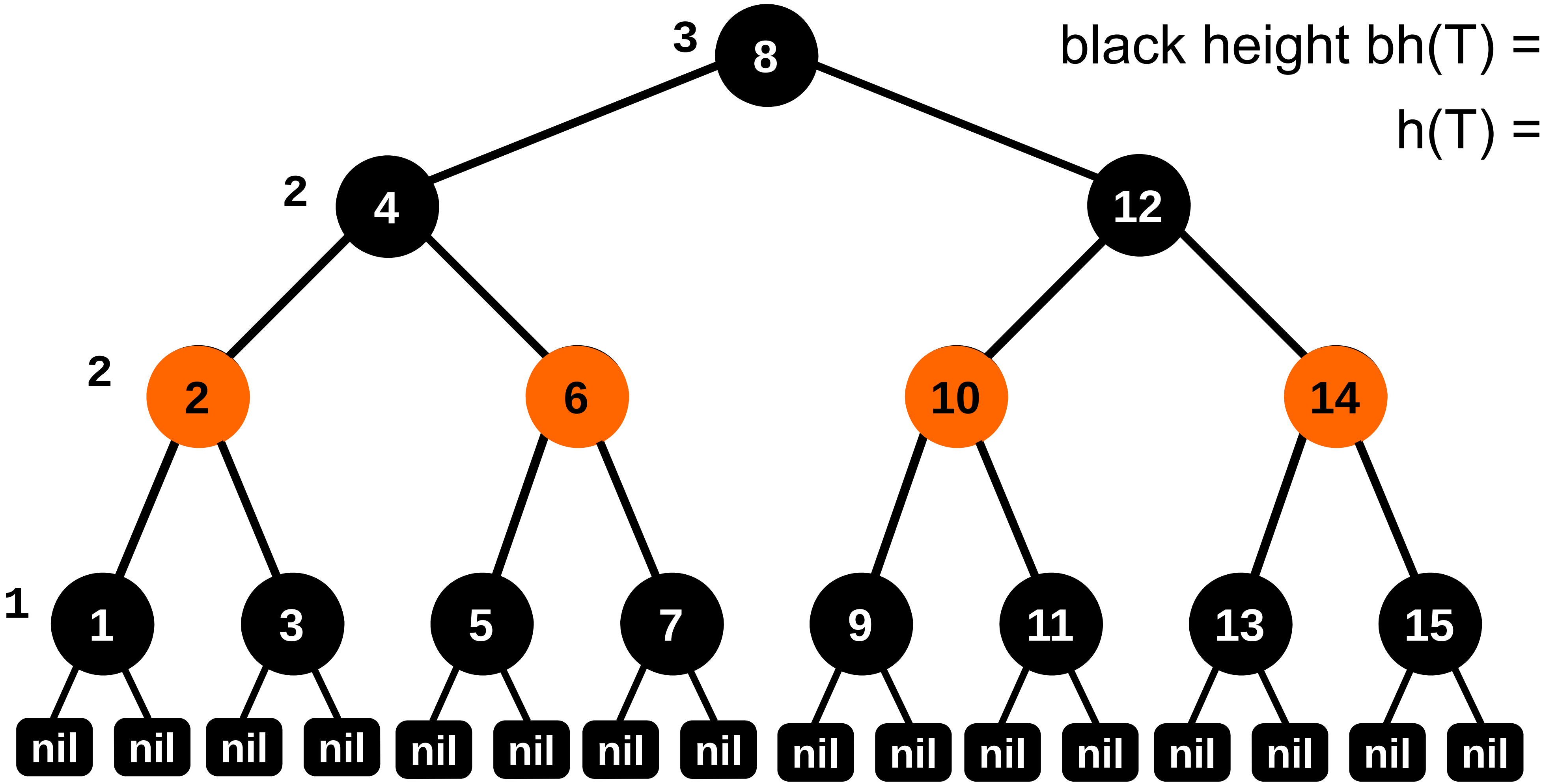
Binary Search Tree -> RB Tree



black height bh(T) = 4

h(T) = 4

Binary Search Tree -> RB Tree



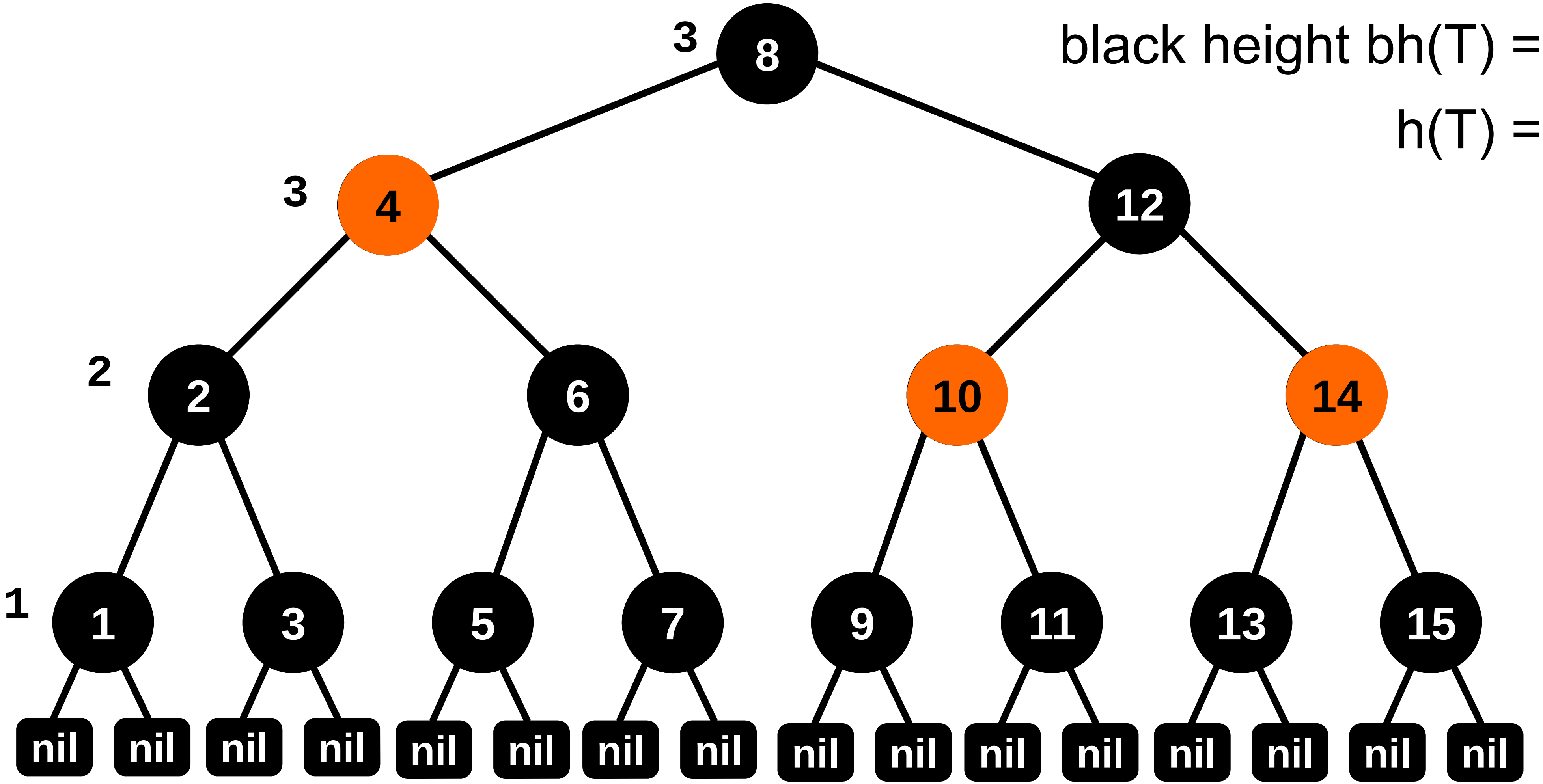
black height bh(T) = 3

h(T) = 4

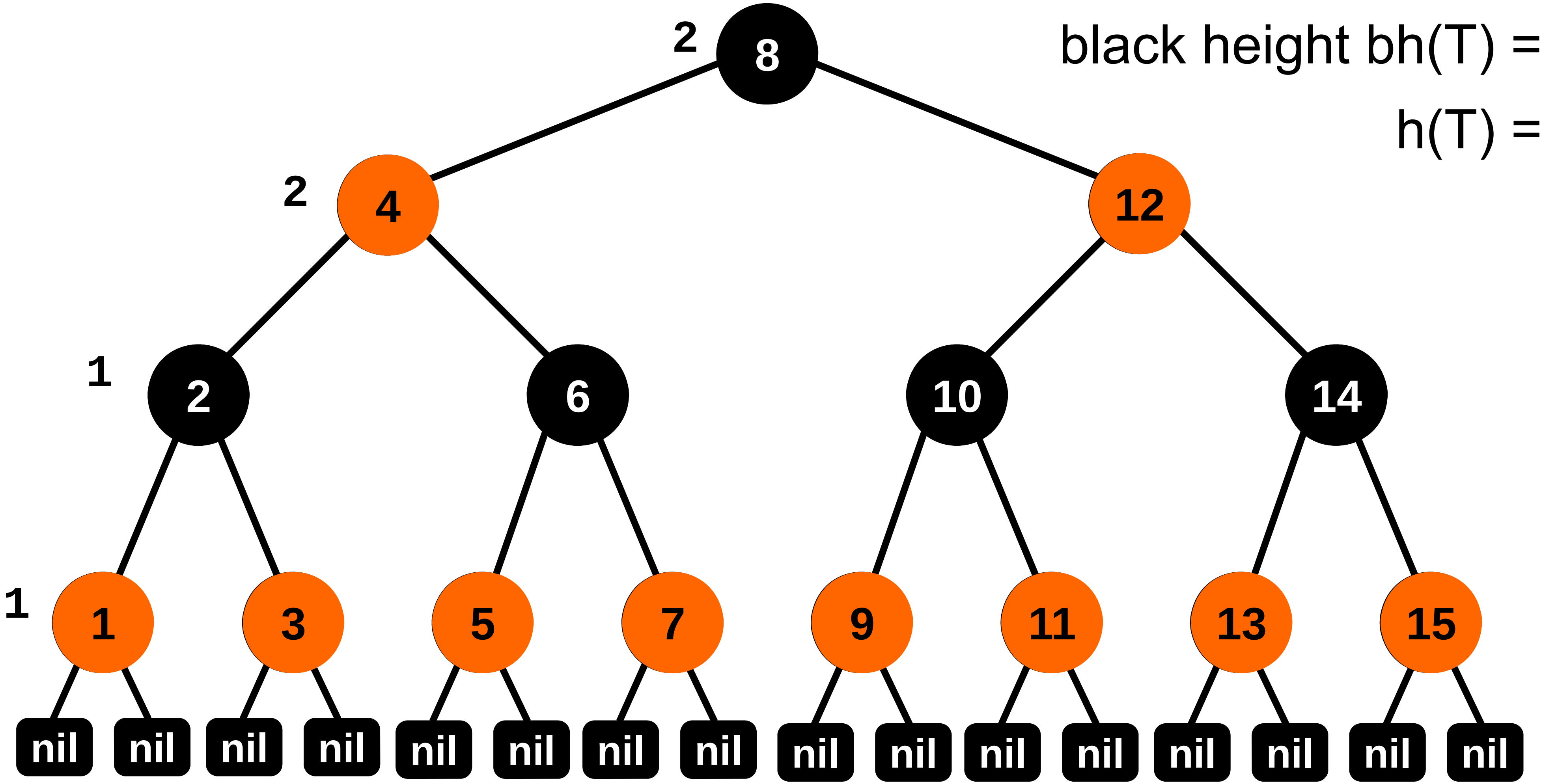
Binary Search Tree -> RB Tree

black height bh(T) = 3

h(T) = 4



Binary Search Tree -> RB Tree



black height bh(T) = 2

h(T) = 4

Red-Black tree

Black-height *bh*(*x*) of a node **x**

- is the number of black nodes on any path from **x** to a leaf, not counting **x**
- is equal for all paths from **x** to a leaf
- For given **h** is **bh(x)** in the range from **h/2** to **h**
  - if ½ of nodes red => bh(x) ≈ ½ h(x), h(x) ≈ 2 lg(n+1)
  - if all nodes black => bh(x) = h(x) = lg(n+1)

Height **h(x)** of a RB-tree rooted in node **x**

- is at maximum twice of the optimal height of a balanced tree
- $h \le 2\lg(n+1)$ …. $h \in \Theta(\lg(n))$

RB-tree height proof [Cormen, p.264]

A red-black tree with n internal nodes has height h at most $2\lg(n+1)$

Proof 1. Show that subtree starting at x contains at least $2^{bh(x)}-1$ *internal nodes*.

By induction on height of x:

I. If x is a *leaf*, then $bh(x) = 0$, $2^{bh(x)}-1 = 0$ internal nodes //… nil node

II. Consider x with height h *and* two children (with height at most h -1)

- x's children black-height is either $bh(x)$ -1 or $bh(x)$ // x is black or red
- Ind. hypothesis: x's children subtree has at least $2^{bh(x)-1}$ -1 internal nodes
- So subtree rooted at x contains at least
  $(2^{bh(x)-1} -1) + (2^{bh(x)-1} -1) + 1 = 2^{bh(x)} - 1$ internal nodes => proved

Proof 2. Let h = height of the tree rooted at x

- min ½ nodes are black on any path to leaf => $bh(x) \geq h / 2$
- Thus, $n \geq 2^{h/2} - 1 \Leftrightarrow n + 1 \geq 2^{h/2} \Leftrightarrow \lg(n+1) \geq h / 2$
- $h \leq 2\lg(n+1)$

RB-tree Search

Search is performed as in simple BST, node colors do not influence the search.

Search in R-B tree with N nodes takes

1. In general -- at most 2*lg(N+1) key comparisons.
2. In best case when keys are generated randomly and uniformly

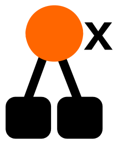
   -- cca 1.002*lg(N) key comparisons,

   very close to the theoretical minimum.

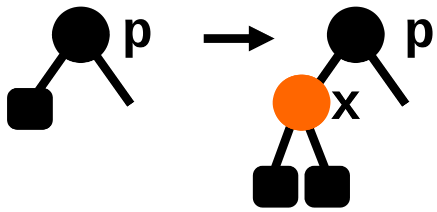
Inserting in Red-Black Tree

Color new node **x** **Red**

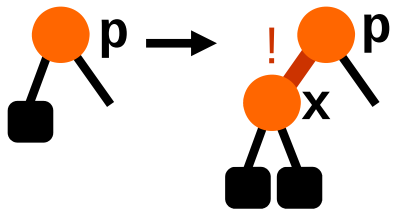
Insert it as in the standard BST



If parent **p** is **Black**, stop. Tree is a Red-Black tree.



If parent **p** is **Red** (3+3 cases)…



resp.

While ***x*** is not root and <u>parent is Red</u>

if ***x***'s *uncle* is Red then case 1 // propagate red up

else { if ***x*** is *Right child* then case 2 // double rotation

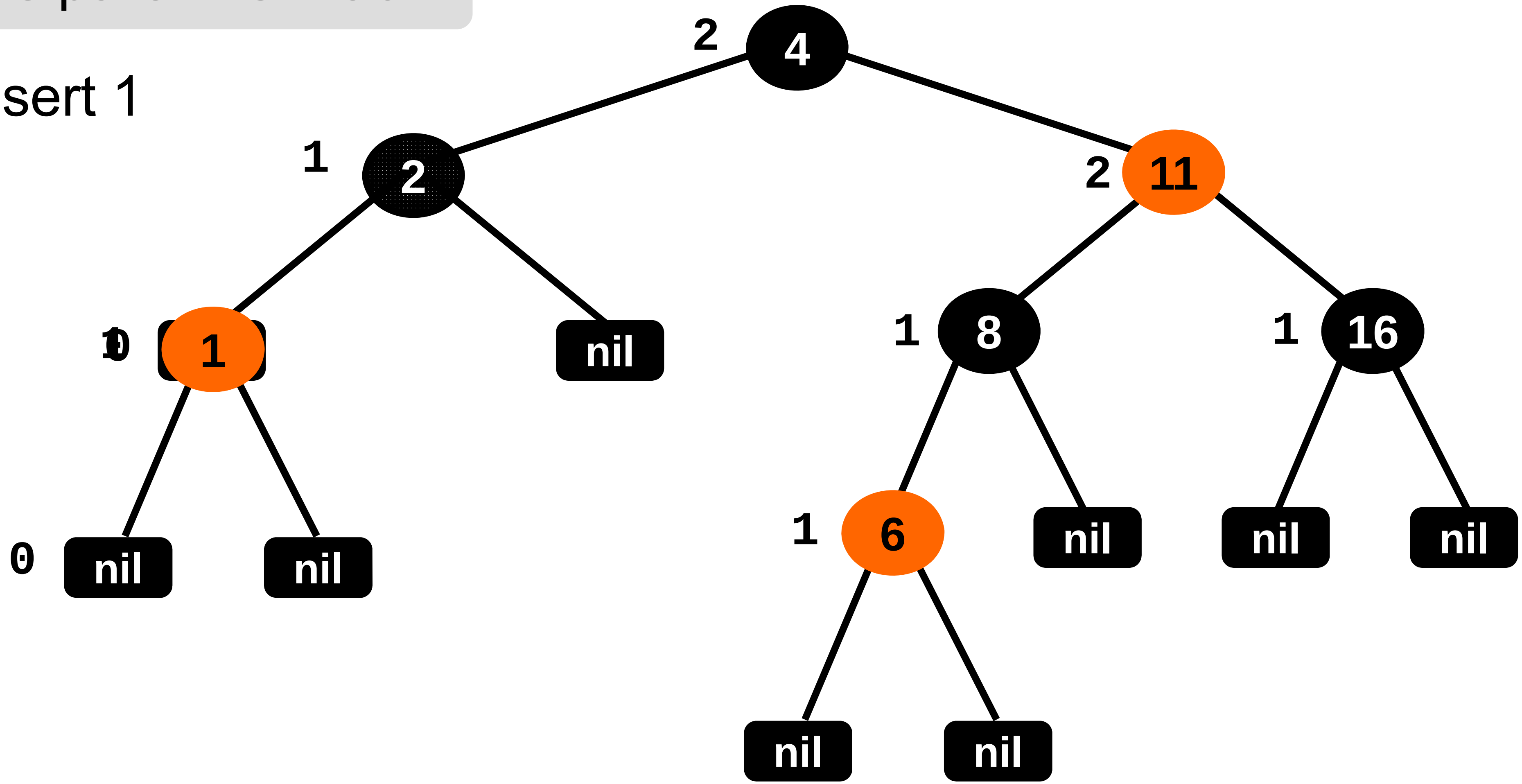
case 3 } // single rotation

Color root Black

Inserting in Red-Black Tree

x's parent is Black

Insert 1



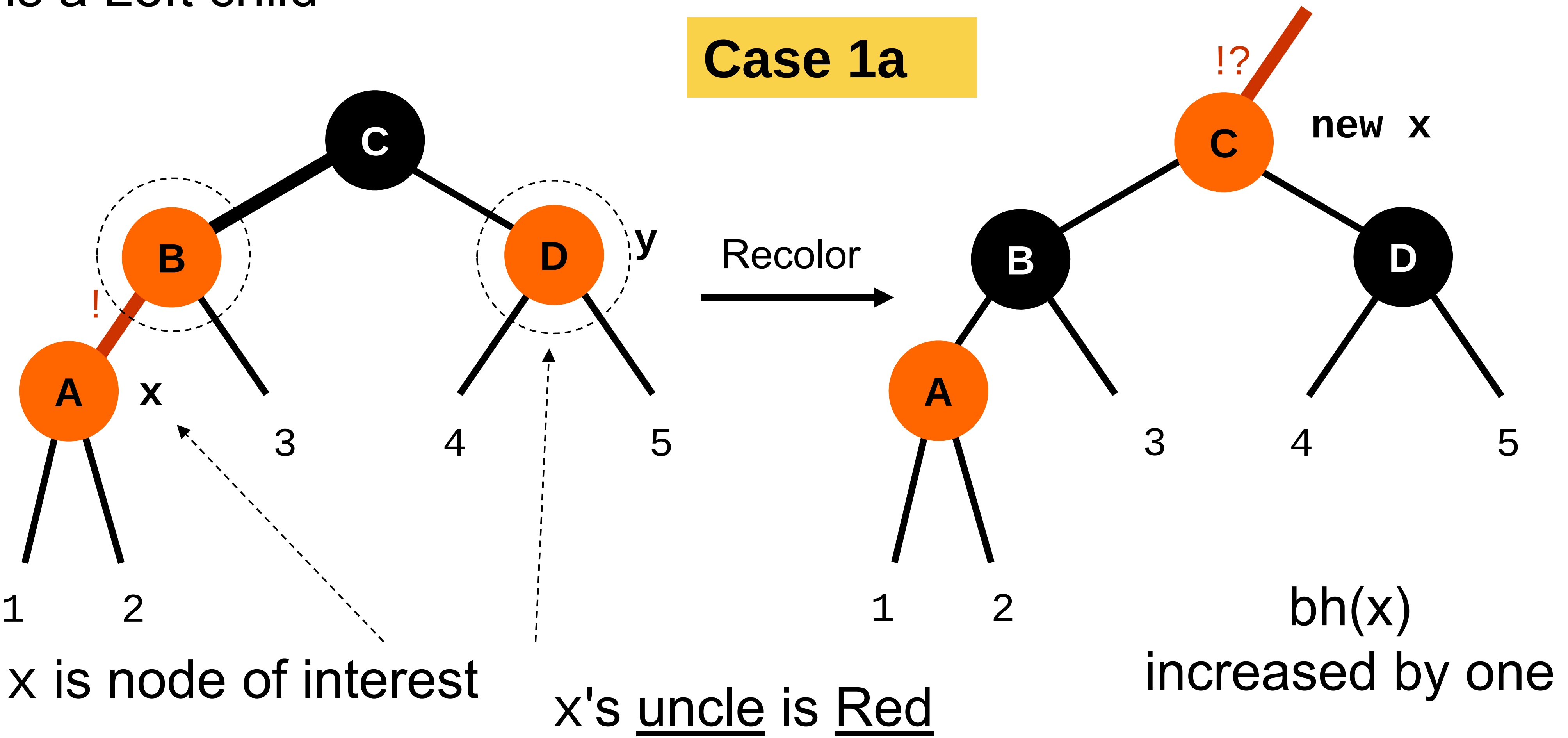
If parent is Black, stop. Tree is a Red-Black tree.

Inserting in Red-Black Tree

x’s parent is Red

x's <u>uncle</u> y is <u>Red</u>

x is a Left child



Inserting in Red-Black Tree

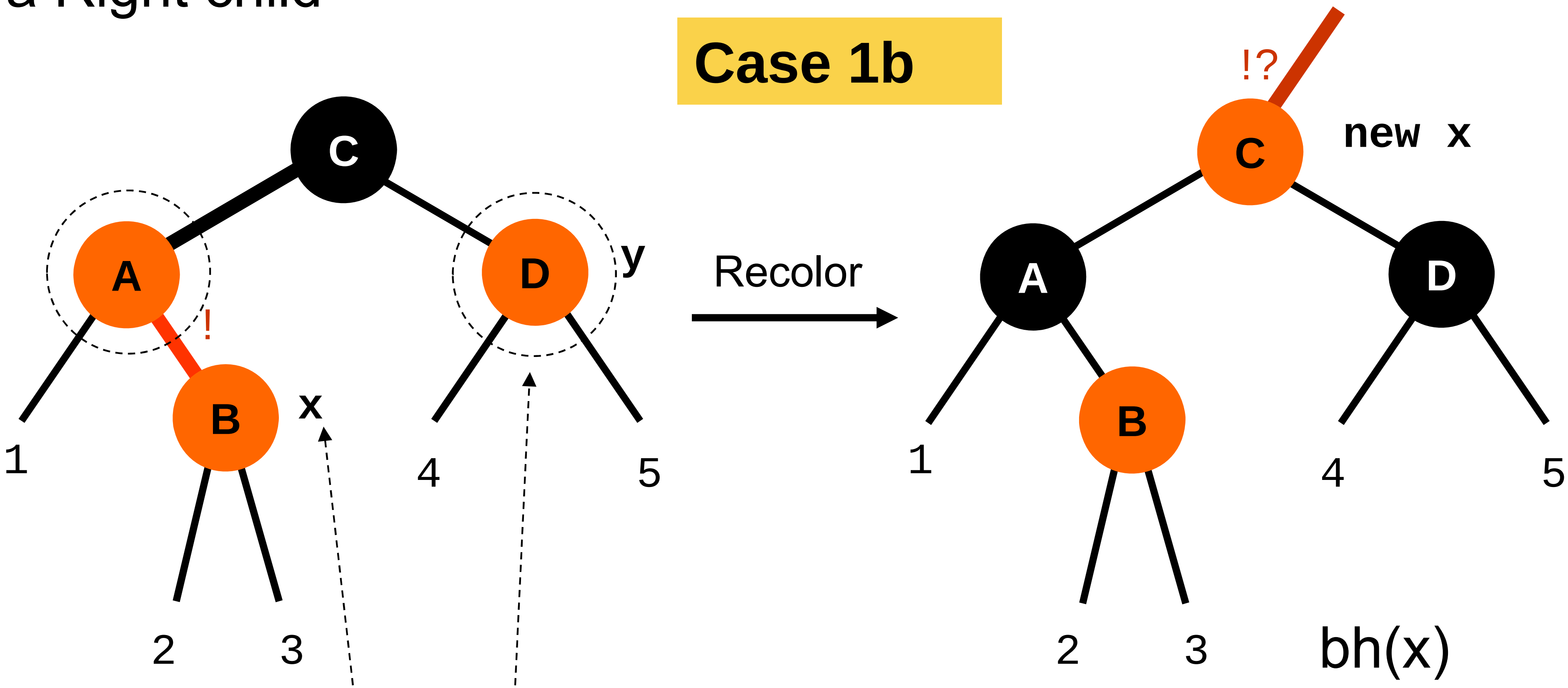
x's parent is Red

x's uncle y is Red

x is a Right child

Loop: x = x.p.p

Case 1b



x is node of interest

x's uncle is Red

bh(x) increased by one

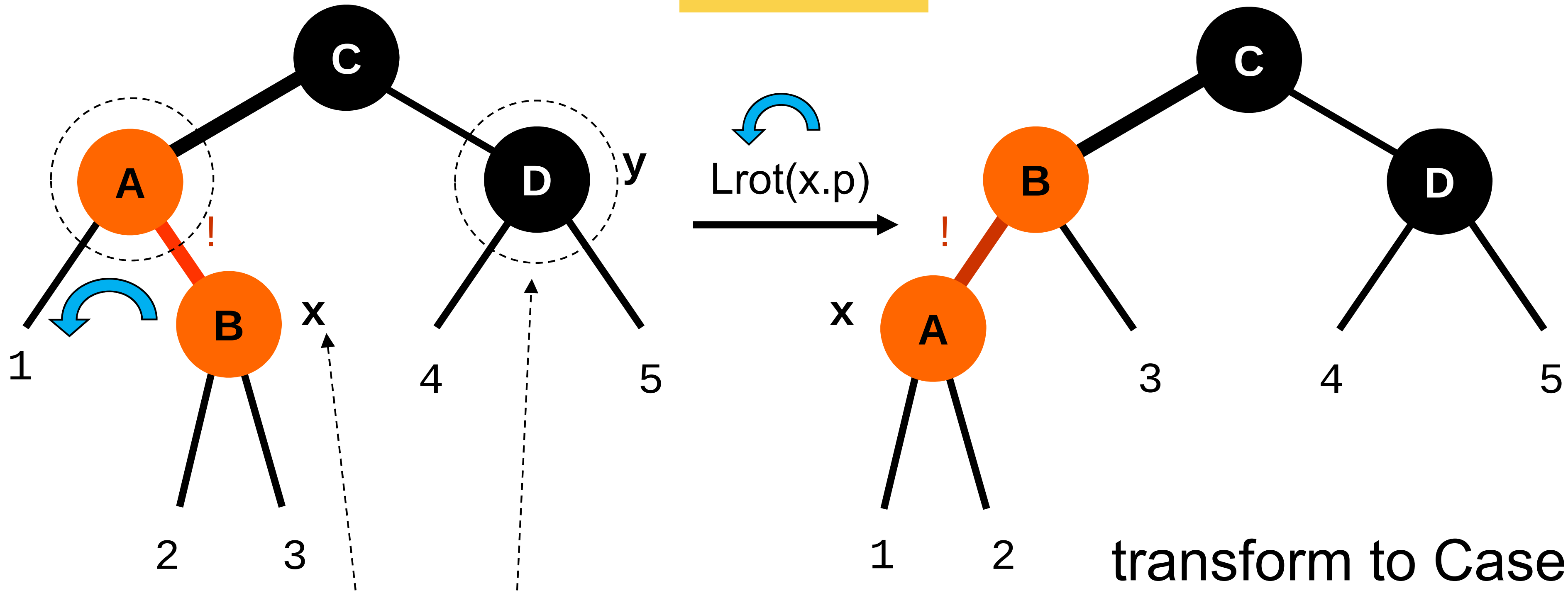
Inserting in Red-Black Tree

x's parent is Red

x's uncle y is Black

x is a Right child

Case 2



transform to Case 3

x is a Right child

x's uncle is Black

Inserting in Red-Black Tree

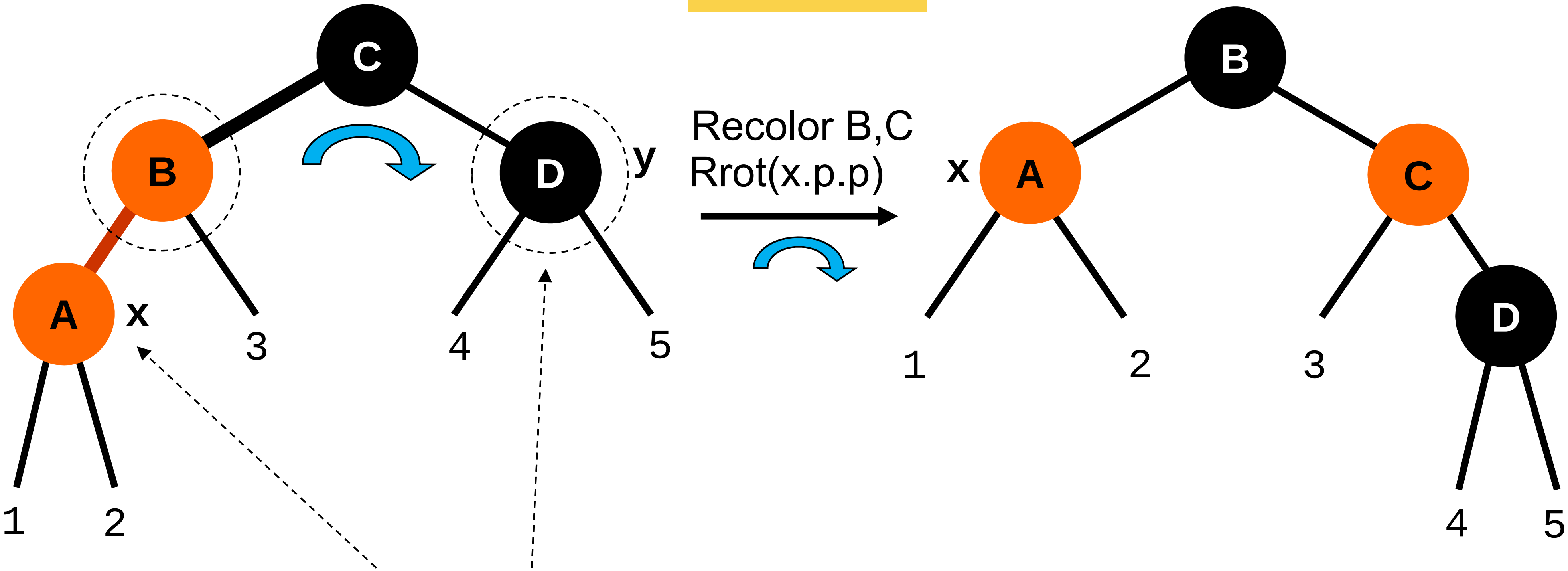
x’s parent is Red

x's uncle y is Black

x is a Left child

Terminal case, tree is a Red-Black tree

Case 3

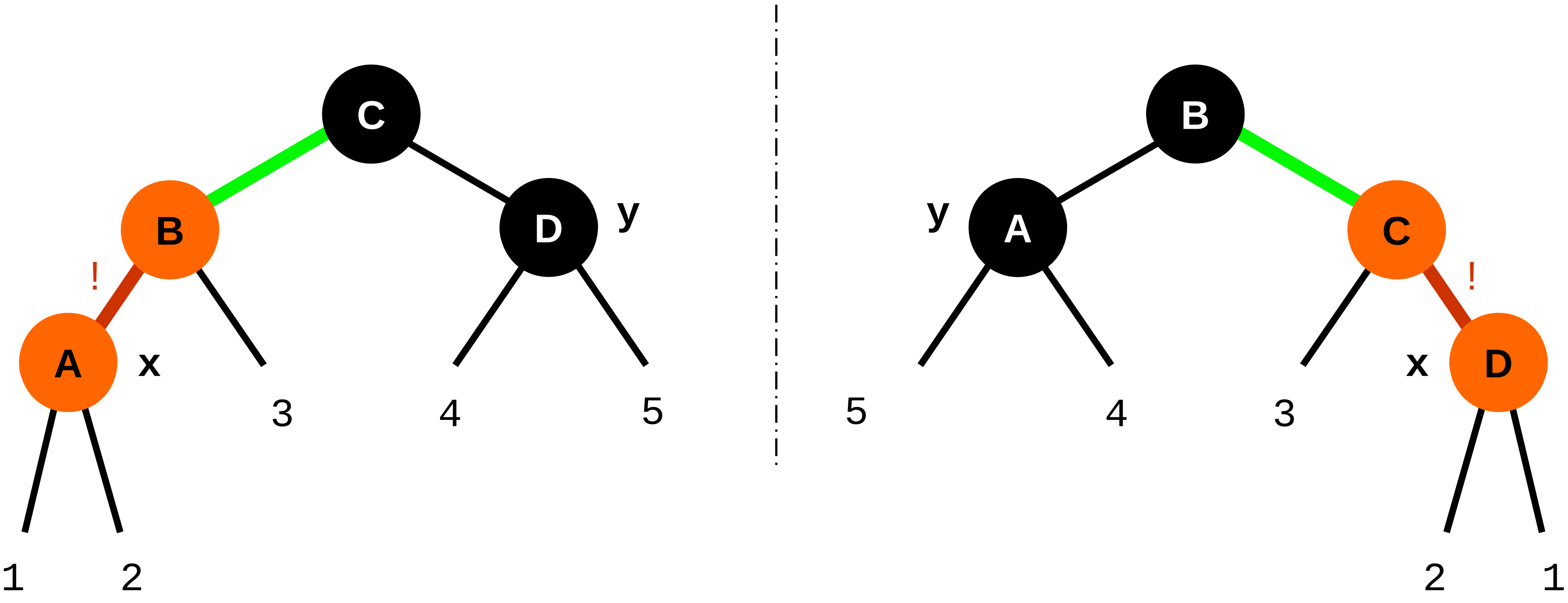


x is a Left child

x's uncle is Black

Inserting in Red-Black Tree

Cases Right from the grandparent
are symmetric



p[x] = parent of x
left[x] = left child of x
y = uncle of x

```
RB-INSERT(T, x)
  TREE-INSERT(T, x)
  color[x] ← RED
  while x ≠ root[T] and color[p[x]] = RED
      do if p[x] = left[p[p[x]]]
            then y ← right[p[p[x]]]
                 if color[y] = RED
                    then color[p[x]] ← BLACK              ▷ Case 1
                         color[y] ← BLACK                 ▷ Case 1
                         color[p[p[x]]] ← RED             ▷ Case 1
                         x ← p[p[x]]                      ▷ Case 1
                    else if x = right[p[x]]
                            then x ← p[x]                 ▷ Case 2
                                 LEFT-ROTATE(T, x)        ▷ Case 2
                         color[p[x]] ← BLACK              ▷ Case 3
                         color[p[p[x]]] ← RED             ▷ Case 3
                         RIGHT-ROTATE(T, p[p[x]])         ▷ Case 3
            else (same as then clause
                    with "right" and "left" exchanged)
  color[root[T]] ← BLACK
```

Red uncle y ->recolor up

[Cormen90]

Inserting in Red-Black Tree

Insertion in $\Theta(\log(n))$ time

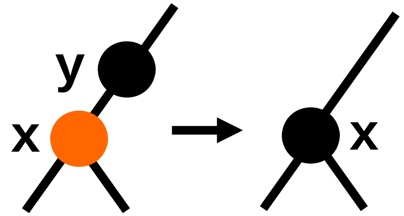
Requires at most two rotations

Deleting in Red-Black Tree

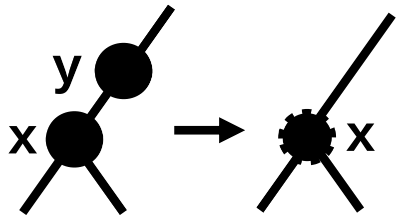
Find node to delete
Delete node as in a regular BST
Node **y** to be physically deleted will have at most one child **x**!!!

If we **delete a Red node**, tree still is a Red-Black tree, **stop**
Assume we **delete a black node**

Let **x** be the left **child of deleted (black) node y**
If **x** is red, color it black and **stop**



while(**x** is not root) AND (**x** is black)
 move **x** with virtual black mark through the tree
 (If **x** is black, mark it virtually double black A)



//note that the whole **x** 's subtree lost 1 unit of black height

Deleting in Red-Black Tree

while(**x** is not root) AND (**x** is black) {

// move **x** with virtual black mark **A** through the tree

// just recolor or rotate other subtree up (decrease bh in R subtree)

if (sibling is **red**)

-> Case 1: Rotate right subtree up, color sibling black, and continue in left subtree with the new sibling

if (sibling is **black** with **both black children**)

-> Case 2: Color sibling red and go up

else // black sibling with **one or two** **red** children

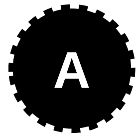
if(**red** left child) -> Case 3: rotate to surface

Case 4: Rotate right subtree up

}

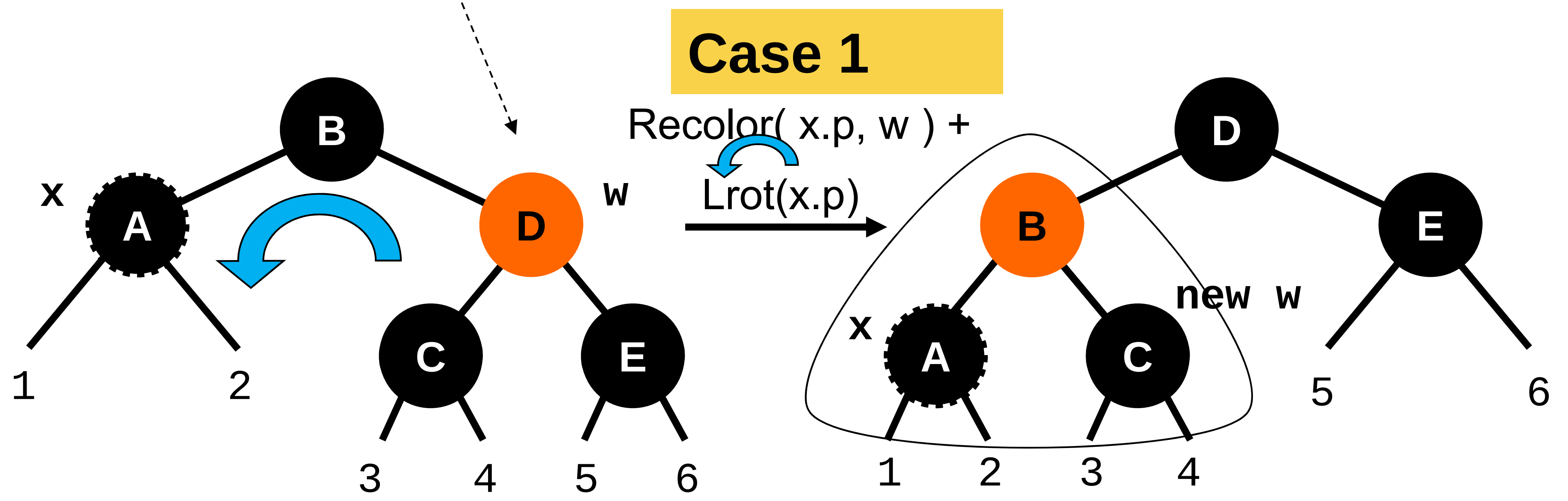
Deleting in R-B Tree - Case 1

x is the child of the physically deleted black node => double black



x's <u>sibling</u> **w** is <u>red</u>

(**x**'s parent *must* be black)



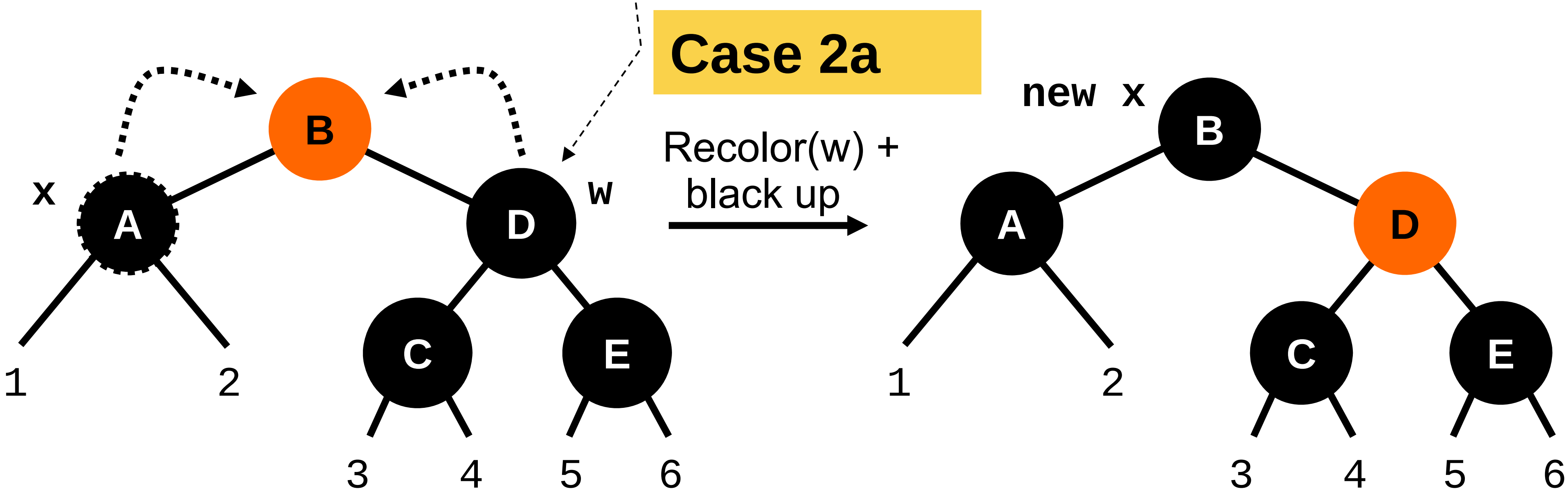
x stays at the same black height

continue

[Possibly transforms to case 2a and terminates – depends on 3,4]

Deleting in R-B Tree - Case 2a

x's sibling **w** is black
x's parent is red
x's sibling left child is black
x's sibling right child is black

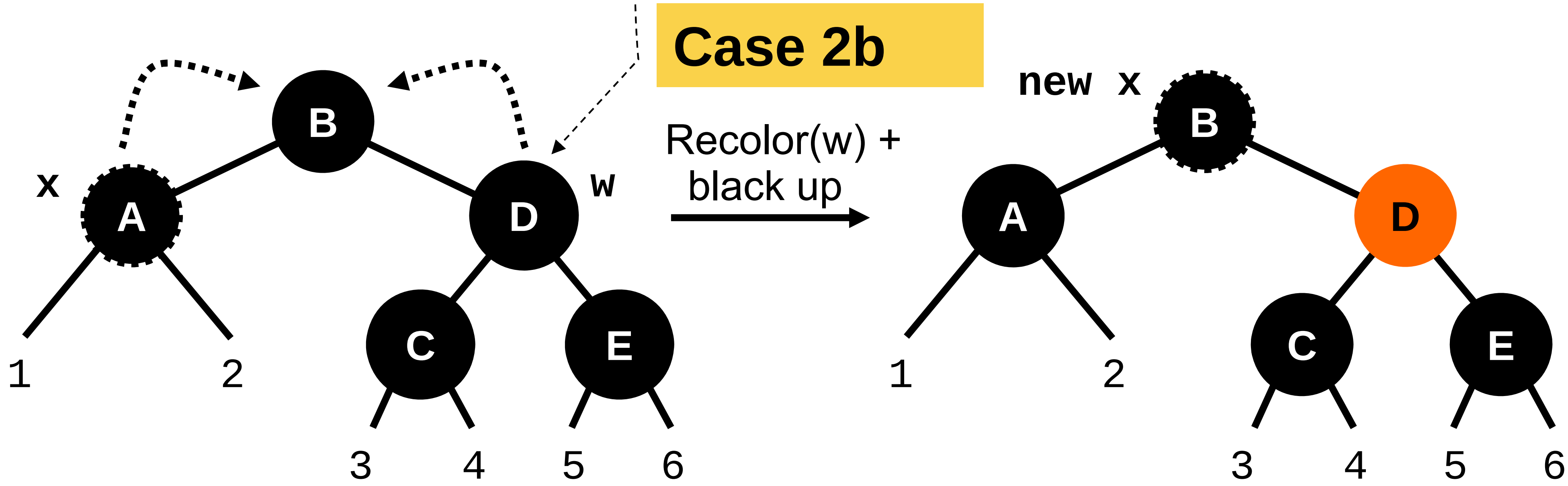


Terminal case, tree is Red-Black tree **STOP**

Note that A's subtree *had* less by 1 black height than D's subtree

Deleting in R-B Tree - Case 2b

x’s sibling **w** is black
x’s parent is black
x’s sibling left child is black
x’s sibling right child is black



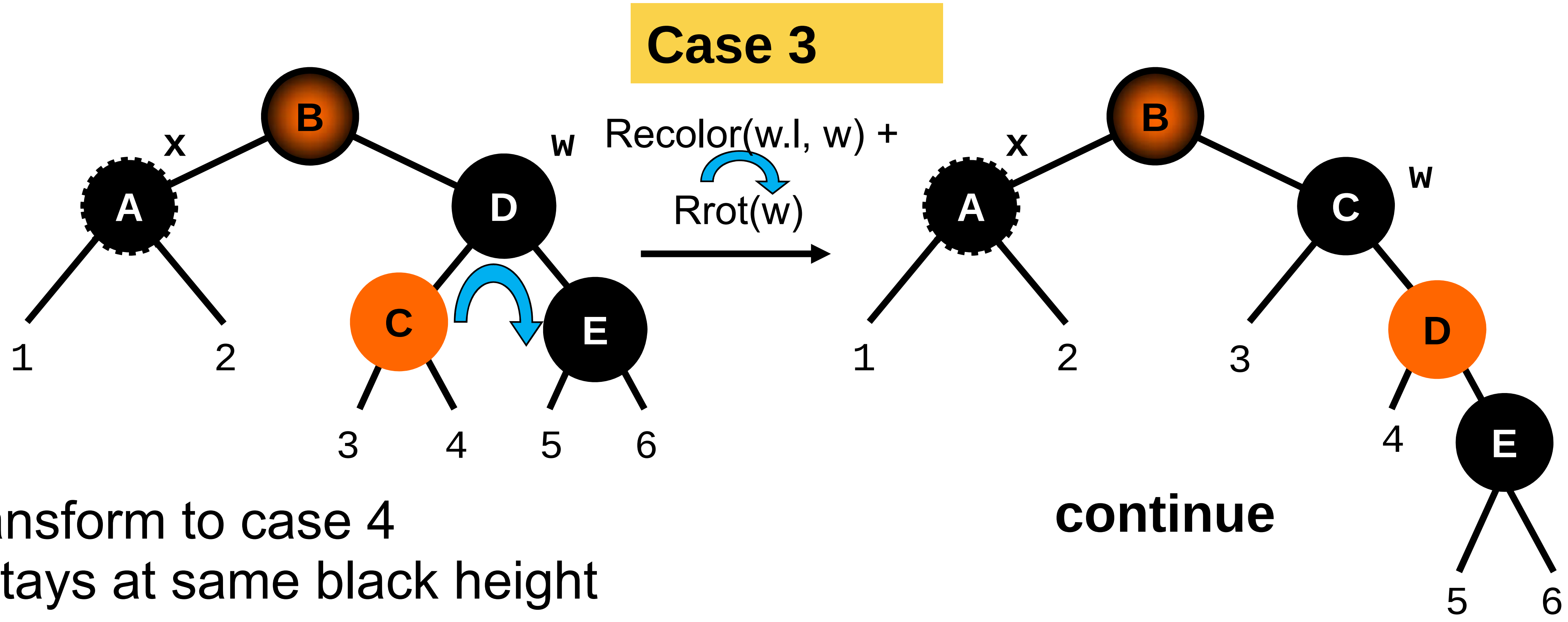
Decreases **x** black height by one

continue with new x

Note that A's subtree *had* less by 1 black height than D's subtree

Deleting in R-B Tree - Case 3

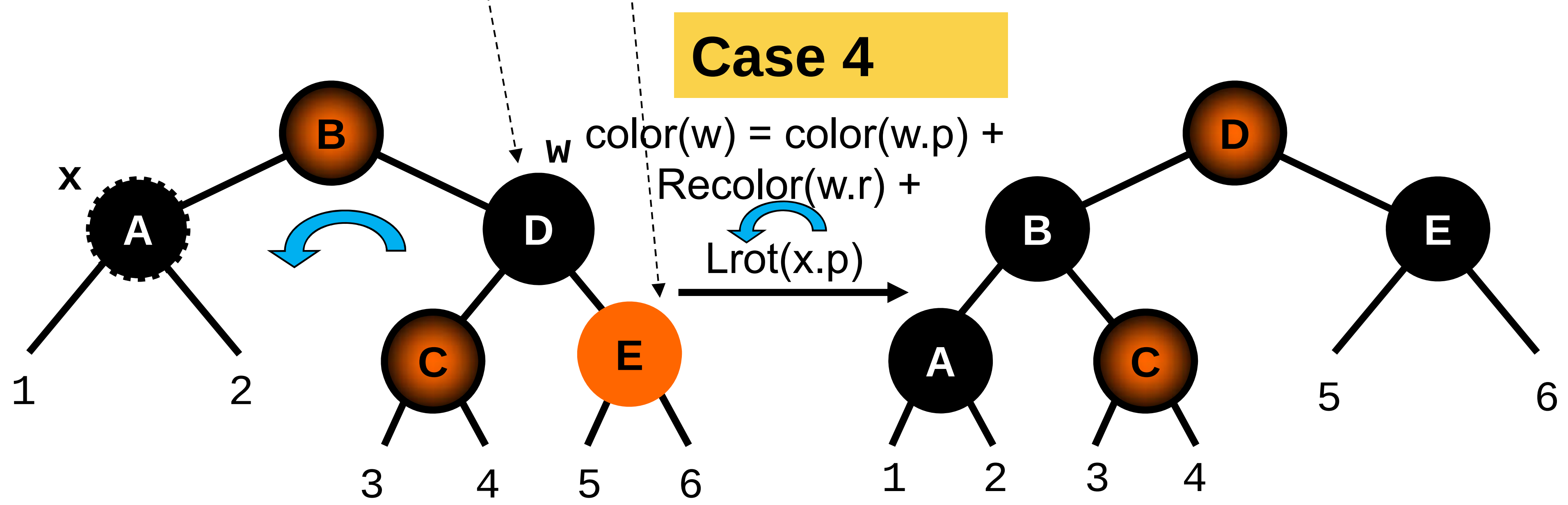
x's sibling **w** is black
x's parent is either
x's sibling left child is red // impossible to color **w** red
x's sibling right child is black



Transform to case 4
x stays at same black height

Deleting in R-B Tree - Case 4

x's sibling w is black
x's parent is either
x's sibling left child is either
x's sibling right child is red // impossible to color w red



Terminal case, tree is Red-Black tree
(w?: D inherits the color of B)

STOP

Deleting in Red-Black Tree

Notation similar to AVL

z = *logically* removed

y = *physically* removed

x = y's only child

```
RB-DELETE(T, z)
if left[z] = nil[T] or right[z] = nil[T]
    then y ← z
    else  y ← TREE-SUCCESSOR(z)
if left[y] ≠ nil[T]
    then x ← left[y]
    else  x ← right[y]
p[x] ← p[y]
if p[y] = nil[T]
    then root[T] ← x
    else  if y = left[p[y]]
              then left[p[y]] ← x
              else  right[p[y]] ← x
if y ≠ z
    then key[z] ← key[y]
           ▷ If y has other fields, copy them, too.
if color[y] = BLACK
    then RB-DELETE-FIXUP(T, x)
return y
```

[Cormen90]

RB-DELETE-FIXUP(T, x)

x = *child* of removed node
p[x] = parent of x
w = sibling of x

```
while x ≠ root[T] and color[x] = BLACK
    do if x = left[p[x]]
          then w ← right[p[x]]
               if color[w] = RED
                  then color[w] ← BLACK                          ▷ Case 1
                       color[p[x]] ← RED                         ▷ Case 1
                       LEFT-ROTATE(T, p[x])                      ▷ Case 1
                       w ← right[p[x]]                           ▷ Case 1
               if color[left[w]] = BLACK and color[right[w]] = BLACK
                  then color[w] ← RED                            ▷ Case 2
                       x ← p[x]                                  ▷ Case 2
                  else if color[right[w]] = BLACK
                          then color[left[w]] ← BLACK            ▷ Case 3
                               color[w] ← RED                    ▷ Case 3
                               RIGHT-ROTATE(T, w)                ▷ Case 3
                               w ← right[p[x]]                   ▷ Case 3
                        color[w] ← color[p[x]]                   ▷ Case 4
                        color[p[x]] ← BLACK                      ▷ Case 4
                        color[right[w]] ← BLACK                  ▷ Case 4
                        LEFT-ROTATE(T, p[x])                     ▷ Case 4
                        x ← root[T]                              ▷ Case 4
          else (same as then clause
                  with "right" and "left" exchanged)
color[x] ← BLACK
```

Case 1: R subtree up, Check L

Case 2: Recolor, Black up, Go up

Case 3: inner R-subtree up

Case 4: R subtree up, stop

[Cormen90]

Deleting in R-B Tree

Delete time is $\Theta(\log(n))$
At most three rotations are done

Which BS tree is the best? [Pfaff 2004]

It is data dependent

- For random sequences

  => use *unsorted tree*, no waste time for rebalancing

- For mostly random ordering with occasional runs of sorted order

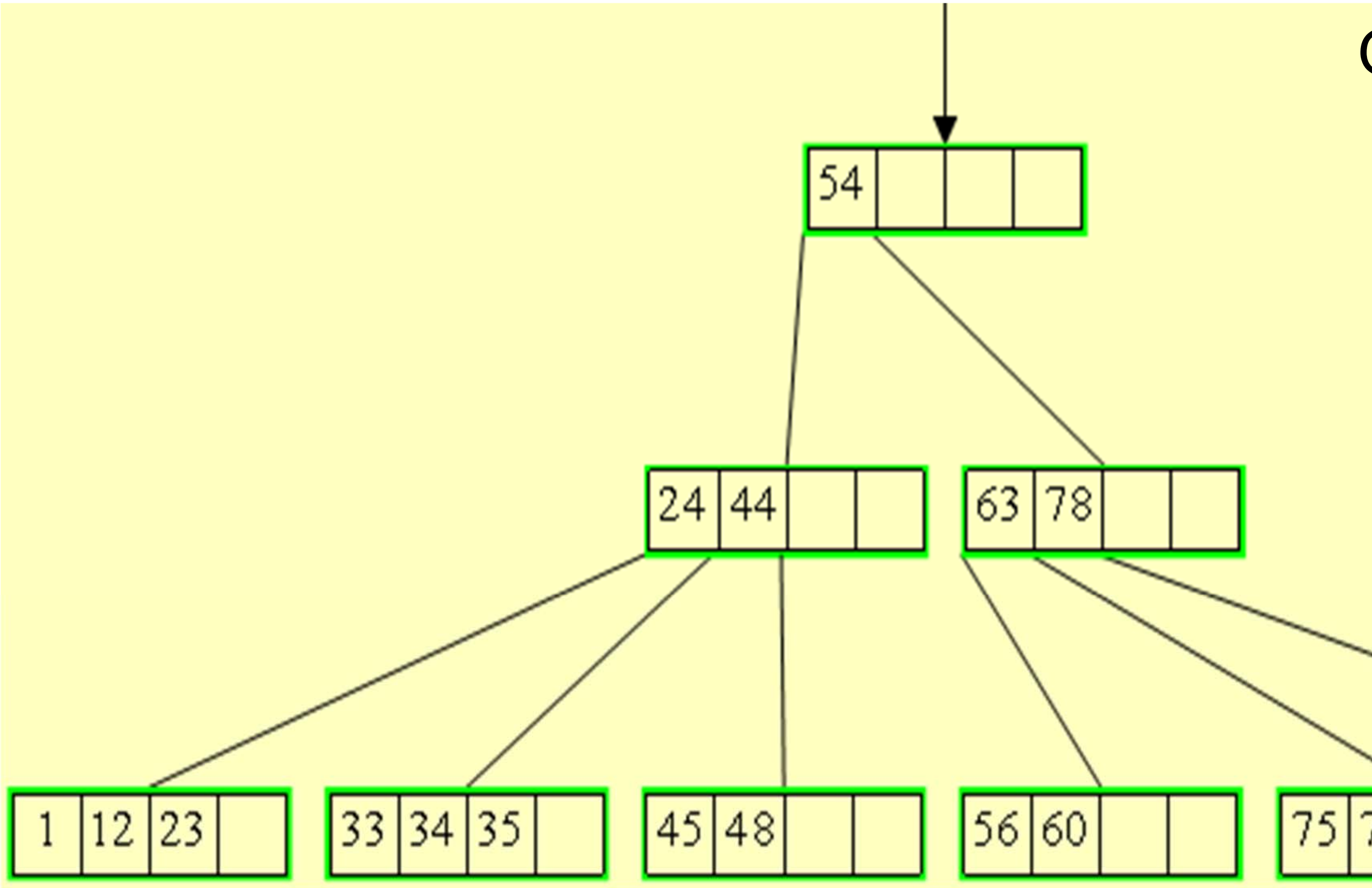
  => use *red-black trees*

- For insertions often in a sorted order and
  - later accesses tend to be random => AVL trees
  - later accesses are sequential or clustered => splay trees
    - self adjusting trees,
    - update each search by moving searched element to the root

B-tree

B-tree as BST on disk

B-tree



Order 5 (5-ary tree)

Min degree t = 3

Based on [Cormen] and [Maire]

B-tree

1. Motivation
2. Multiway search tree
3. B-tree
4. Search
5. Insert
6. Delete

B-tree

Motivation

- Large data do not fit into operational memory -> disk
- Time for disk access is limited by HW (Disk access = Disk-Read, Disk-Write)

DISK : 16 ms
Seek 8ms + rotational delay 7200rpm 8ms

Instruction:
800 MHz 1,25ns

- Disk access is MUCH slower compared to instruction
  - 1 disk access ~ 13 000 000 instructions!!!!
  - Number of disk accesses dominates the computational time

B-tree

Motivation

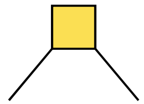
Disk access = Disk-Read, Disk-Write

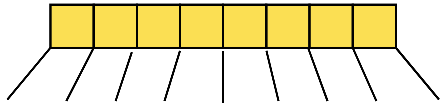
- Disk divided into blocks (512, 2048, 4096, 8192 bytes)
- Whole block transferred

- Design a *multiway search tree*
- Each node fits to one disk block

B-tree

Multiway search tree

= a generalization of Binary search tree  (*m*=2)

Each node has at most *m* children  (*m*>2)

Internal node with *n* keys has *n*+1 successors, $n < m$
(except root)

Leaf nodes with no successors

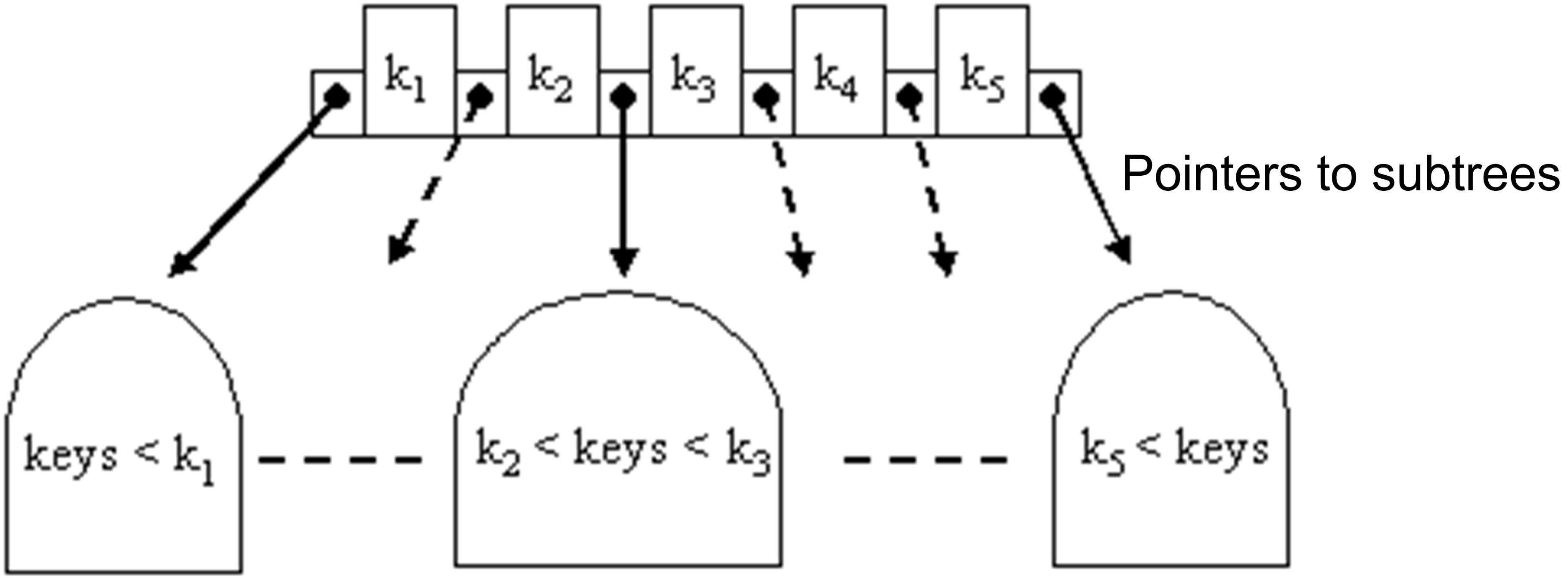
Tree is ordered

Keys in nodes separates the ranges in subtrees

B-tree

Multiway search tree – internal node

Keys in internal node separate the ranges of keys in subtrees



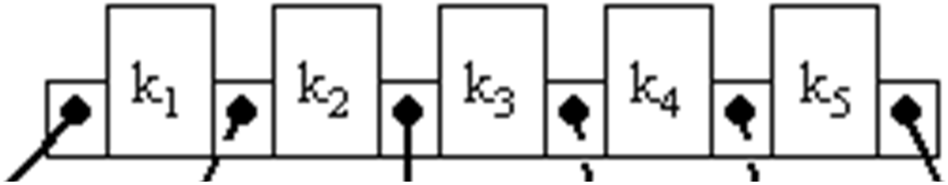
© Frederic Maire, QUT

$$k_1 < k_2 < \ldots < k_5$$

B-tree

Multiway search tree – leaf node

Leaves have no subtrees and do not use pointers



Leaves have no pointers to subtrees

$$k_1 < k_2 < \ldots < k_5$$

B-tree

B-tree

= of order *m* is an *m*-way search tree, such that

- All leaves have the same height (B-tree is balanced)
- All internal nodes are constrained to have
  - at least *m*/2 non-empty children and (precisely later)
  - at most *m* non-empty children
- The root can have 0 or between 2 to *m* children
  - 0 - leaf
  - *m* - a **full node**

B-tree

B-tree – problems with notation

Different authors use different names

- Order *m* B-tree
  - Maximal number of children
  - Maximal number of keys   (No. of children - 1)
  - Minimal number of keys
- Minimum degree *t*
  - Minimal number of children [Cormen]

B-tree

B-tree – problems with notation

Relation between minimal and maximal number of children also differs

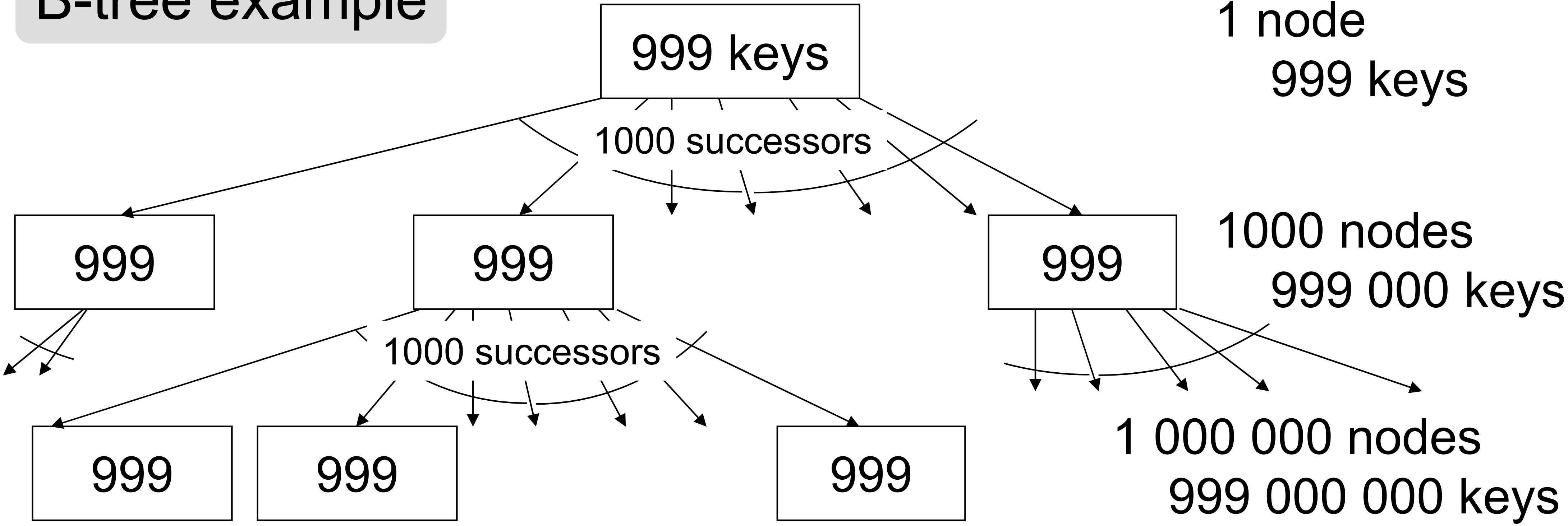
For minimal number *t* of children

Maximal number *m* of children is

- $m = 2t - 1$ simple B-tree, multiphase update strategy
- $m = 2t$ optimized B-tree, singlephase update strategy

B-tree

B-tree example



B-tree of order *m*=1000 of height 2 contains

1 001 001 nodes (1+1000 + 1 000 000)

999 999 999 keys ~ one billion keys (1 miliarda klíčů)

B-tree

B-tree node fields

n … number of keys k_i stored in the node $n < m$.

Node with $n = m$-1 is a **full-node**

k_i … n keys, stored in non-decreasing order

$k_1 \leq k_2 \leq \ldots \leq k_n$

leaf … boolean value, true for leaf, false for internal node

c_i … n+1=m pointers to successors (undefined for leaves)

Keys k_i separate the keys in subtree:

For $keys_i$ in the subtree with root k_i holds

$$keys_1 \leq k_1 \leq keys_2 \leq k_2 \leq \ldots \leq k_n \leq keys_{n+1}$$

B-tree

B-tree algorithms

- Search
- Insert
- Delete

B-tree search

Similar to BST tree search

Keys in nodes sequentially or binary search

Input: pointer to tree root and a key *k*

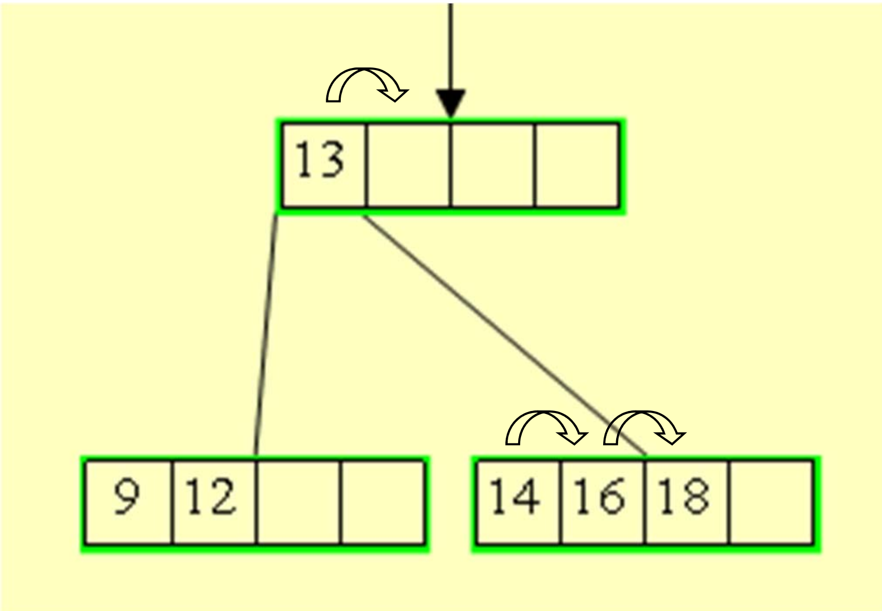
Output: an ordered pair (*y*, *i*), node *y* and index *i*

such that $y.k[i] = k$

or NIL, if *k* not found

B-tree search

Search 17 (17)

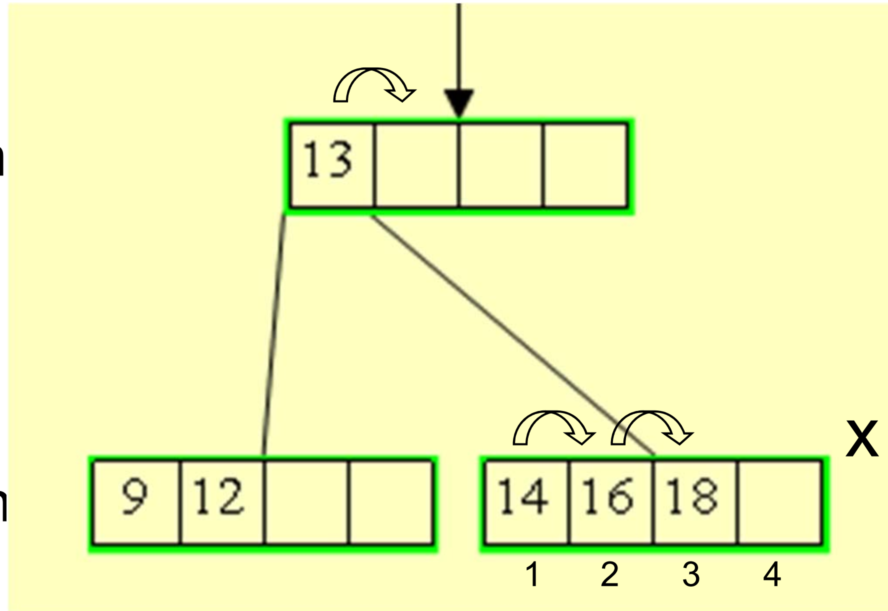


17 not found => return NIL

n

n

Search 18 (18)



18 found => return (x, 3)

B-tree search

```
B-treeSearch(x,k)
   i ← 1
   while i ≤ x.n  and  k > x.k[i]      //sequential search
       do i ← i+1
   if i ≤ x.n and  k = x.k[i]
       return (x, i)                  // pair: node & index
   if x.leaf
       then return NIL
       else
            Disk-Read(x.c[i]) // tree traversal
            return B-treeSearch(x.c[i],k)
```

B-tree search

B-treeSearch complexity

Using tree order *m*

Number of disk pages read is

$$O(h) = O(\log_m n)$$

Where h is tree *height* and

m is the tree order

n is number of tree nodes

Since num. of keys $x.n < m$, the while loop takes $O(m)$ and

total time is $\mathbf{O(m \log_m n)}$

B-tree search

B-treeSearch complexity Using minimum degree t

Number of disk pages read is

$$O(h) = O(\log_t n)$$

Where h is tree *height* and

t is the minimum degree of B-tree

n is number of tree nodes

Since num. of keys $x.n < 2t$, the while loop takes $O(t)$

and

total time is $\mathbf{O(t \log_t n)}$

B-tree update strategies

Two principal strategies

1. Multiphase strategy
   "solve the problem, when appears" m=2t-1 children

2. Single phase strategy [Cormen]
   "avoid the future problems" m =2t children

Actions:

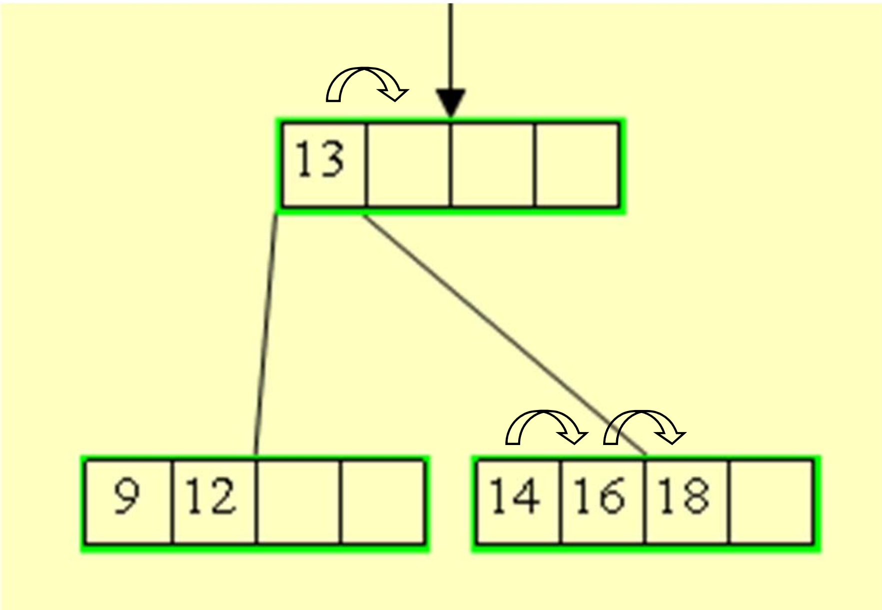
Split full nodes

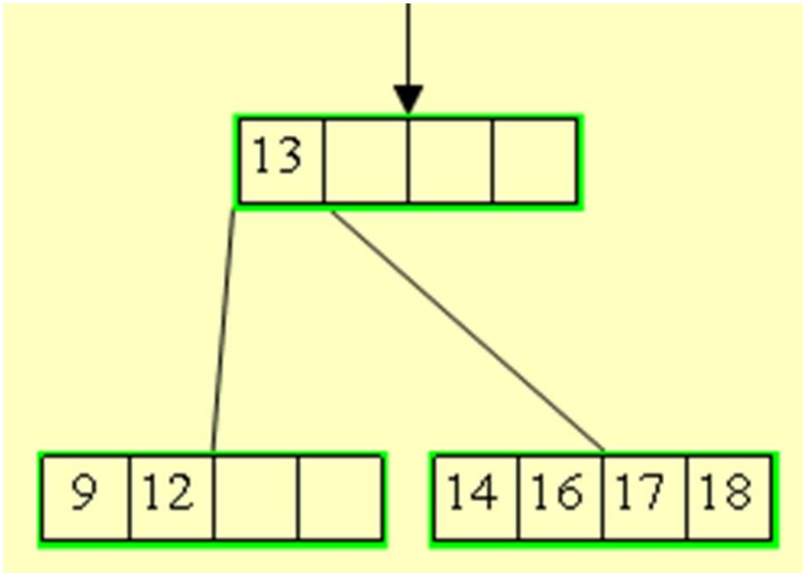
Merge nodes with less than minimum entries

B-tree insert - 1.Multiphase strategy

Insert to a **non-full** node

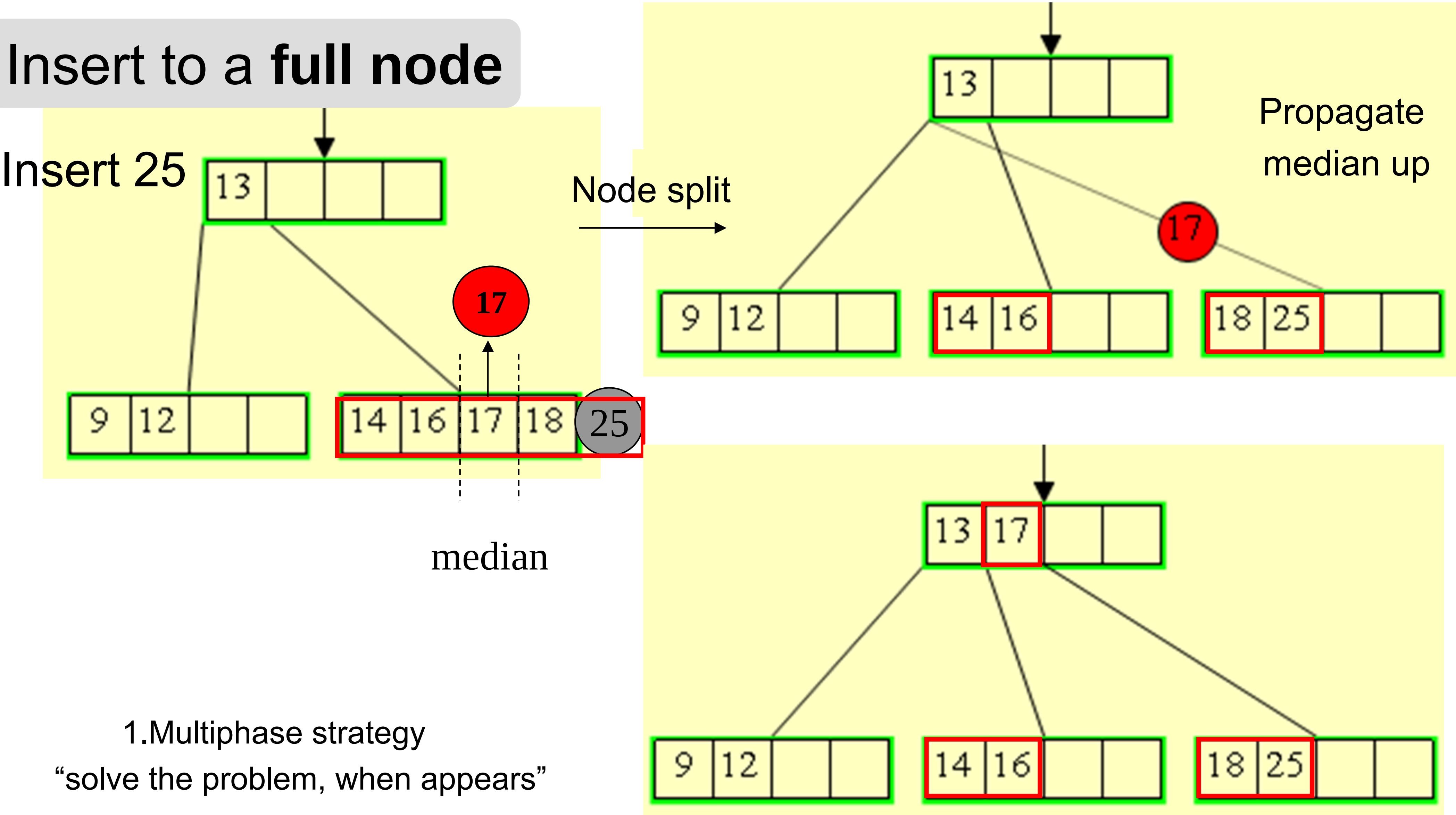
Insert 17





B-tree insert - 1.Multiphase strategy

Insert to a **full node**



1.Multiphase strategy
“solve the problem, when appears”

B-tree insert - 1.Multiphase strategy

Insert (x, T) - pseudocode

x…key, T…tree

Top down phase

Find the leaf for *x*

If not full, insert *x* and stop

Bottom-up phase

while (current_node full) (node overflow)

- find median (in keys in the node after insertion of x)
- split node into two
- promote median up as new *x*
- current_node = parent of current_node or new root

Insert *x* and stop

B-tree insert - 2.Singlephase strategy

Principle: "avoid the future problems"

Top down phase only

- Split the full node with 2t-1 keys when enter
- It creates space for future medians from the children
- No need to go bottom-up

- Splitting of
  - Root => tree grows by one
  - Inner node or leaf => parent gets median key

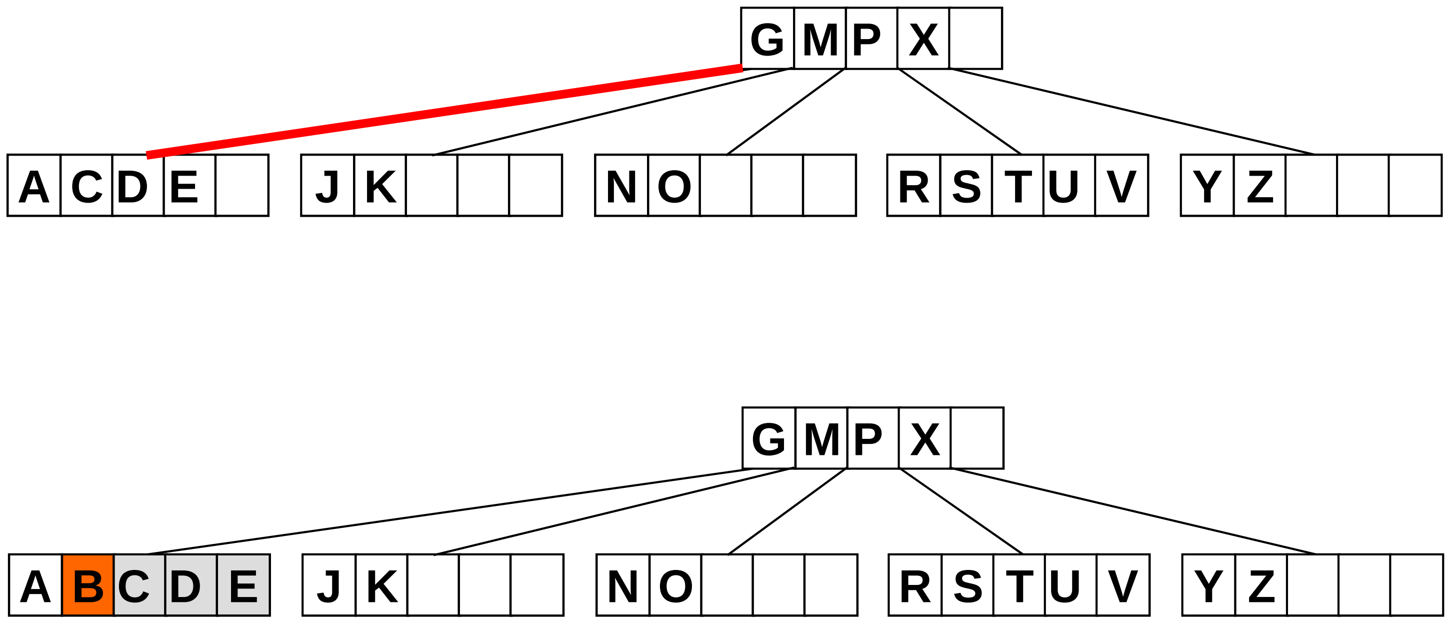
B-tree insert - 2.Singlephase strategy

Insert to a **non-full** node

m = 2*t* = 6 *children*

m-1 keys = odd max number

Insert B

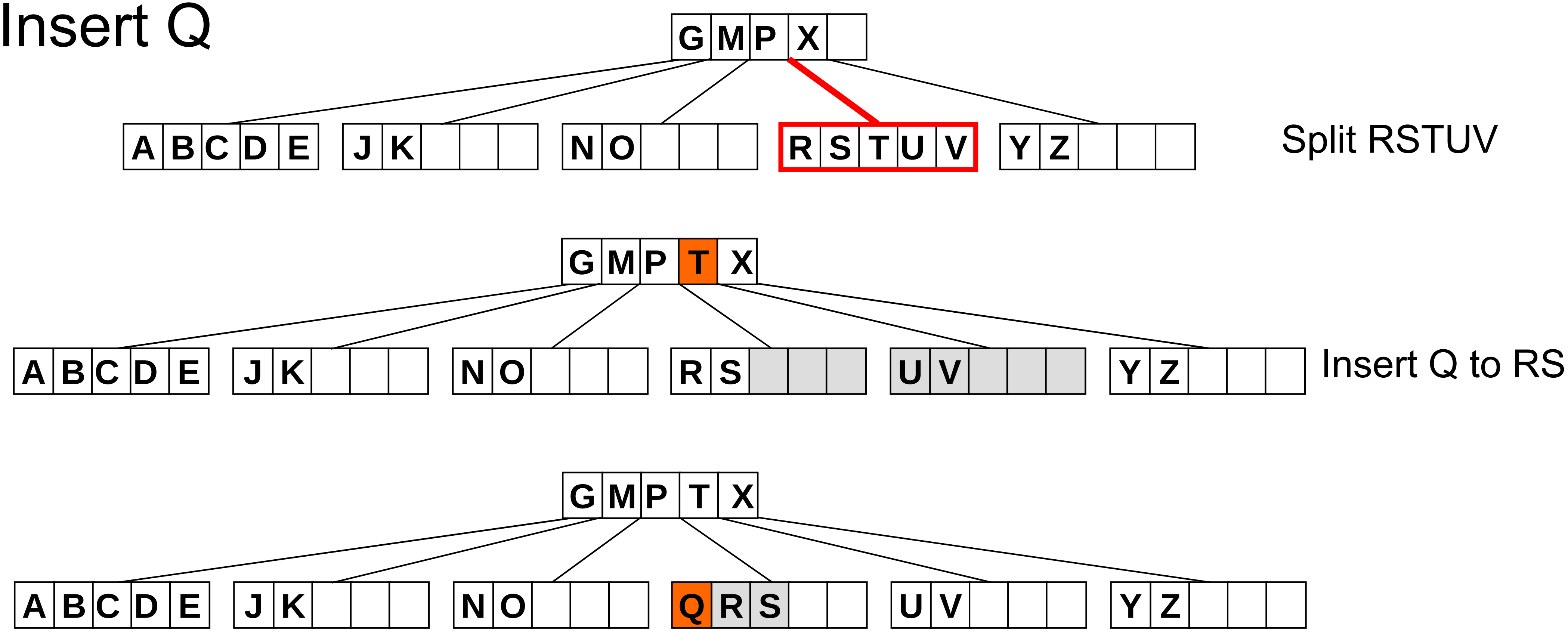


B-tree insert - 2.Singlephase strategy

1 new node

Splitting a passed **full node** and insert to a **not full** node

Insert Q

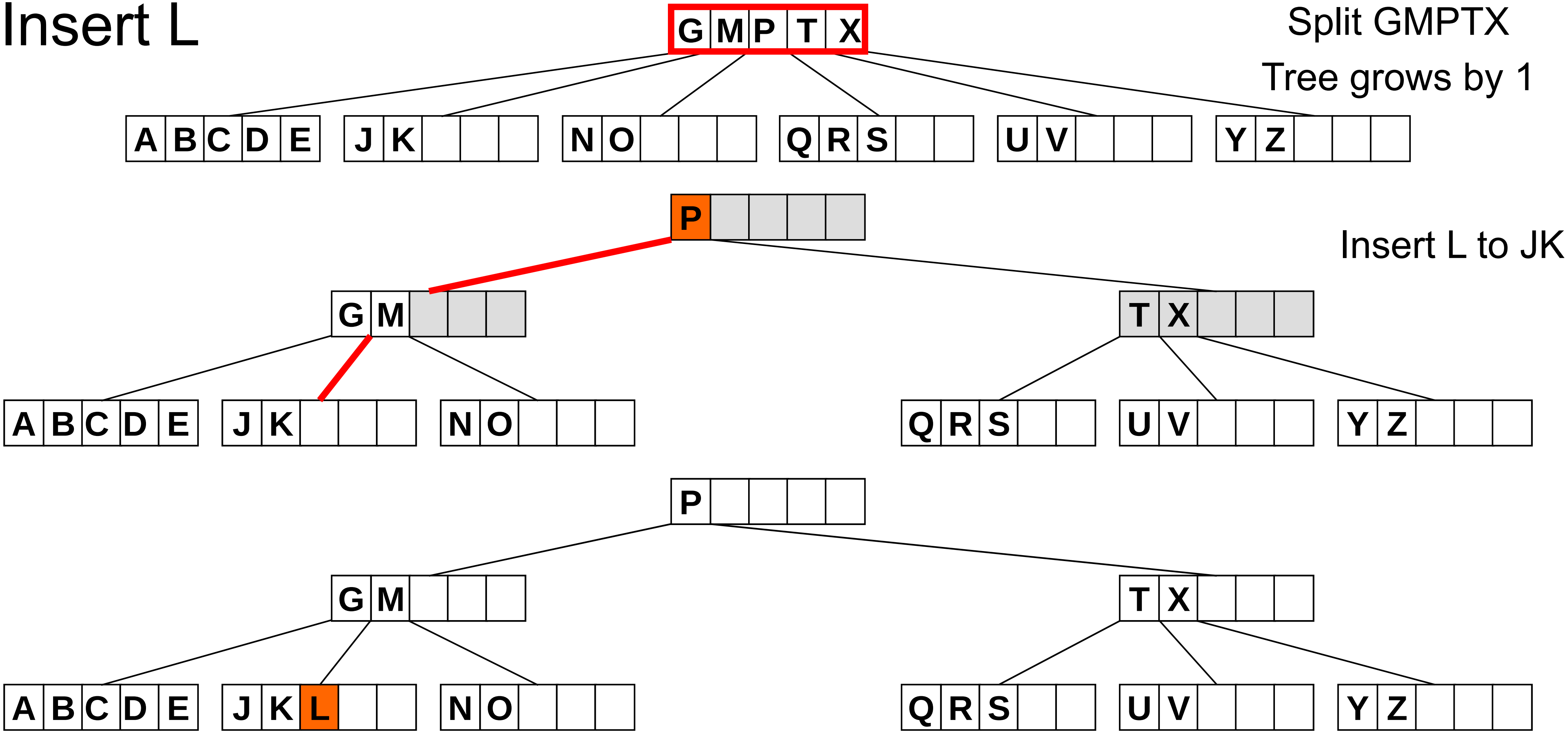


B-tree insert - 2.Singlephase strategy

2 new nodes

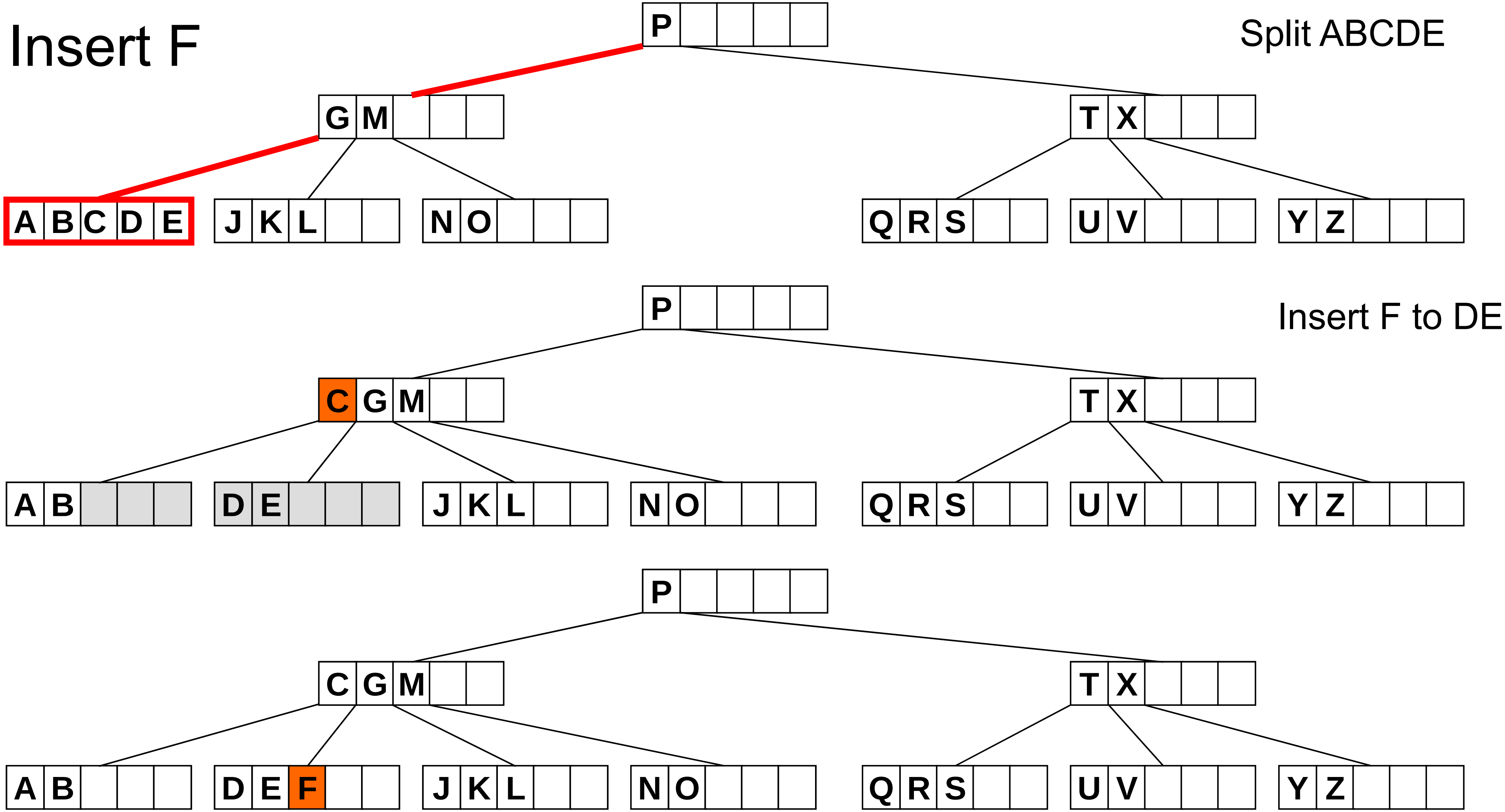
Splitting a passed **full root** and insert to a **not full** node

Insert L



B-tree insert - 2.Singlephase strategy

Insert F



B-tree insert - 2.Singlephase strategy

Insert (x, T) - pseudocode

Top down phase only

x …key, *T*… tree

While searching the leaf *x*

    if (node full)

        find median (in keys in the full node only)

        split node into two

        insert median to parent (there is space)

Insert *x* and stop

B-tree delete

Delete (x, btree) - principles

Multipass strategy only

- Search for value to delete
- Entry is in leaf

  is simple to delete. Do it. Corrections of number of elements later...
- Entry is in Inner node
  - It serves as separator for two subtrees
  - swap it with predecessor(x) or successor(x)
  - and delete in leaf

Leaf in detail

if leaf had more than minimum number of entries

  delete x from the leaf and STOP

else

  redistribute the values to correct and delete x in leaf

  (may move the problem up to the parent,

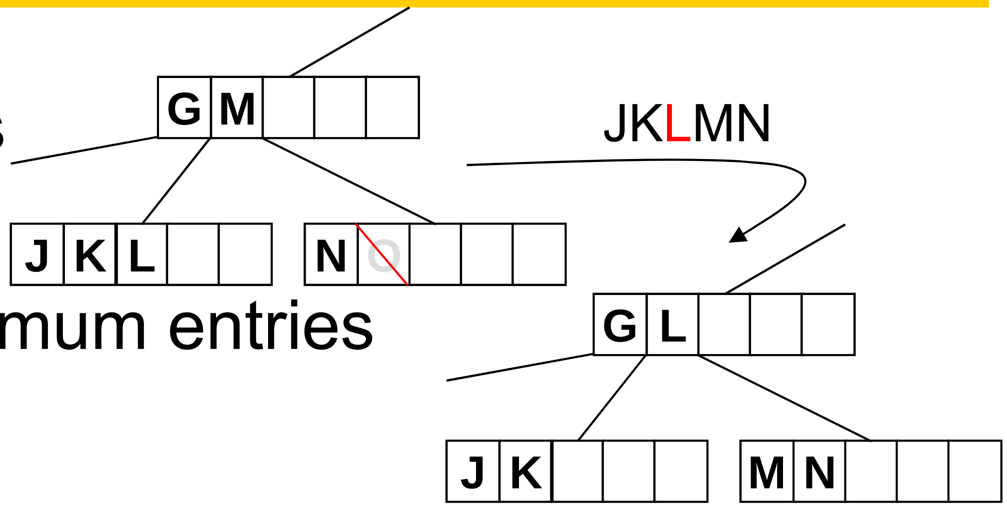
  problem stops by root, as it has no minimum number of entries)

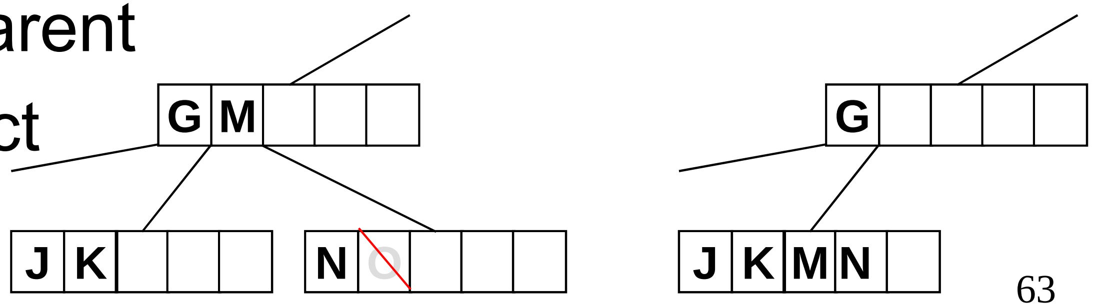
B-tree delete

Node has less than minimum entries

- Look to siblings left and right
- If one of them has more than minimum entries
  - Take some values from it
  - Find new median in the sequence:

    (sibling values – separator- node values)
  - Make new median a separator (store in parent)
- Both siblings are on minimum
  - Collapse node – separator – sibling to one node
  - Remove separator from parent
  - Go up to parent and correct





B-tree delete

Delete (x, btree) - pseudocode

Multipass strategy only

```
if(x to be removed is not in a leaf)
   swap it with successor(x)
currentNode = leaf
while(currentNode underflow)
   try to redistribute entries from an immediate
        sibling into currentNode via its parent
   if(impossible) then merge currentNode with a
        sibling and one entry from the parent
   currentNode = parent of CurrentNode
```

Maximum height of B-tree

$h \leq \log_{\lceil m/2 \rceil} ((n+1)/2)$

half node used for k,
half of children

Gives the upper bound to number of disk accesses

See [Cormen] for details

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