Evolutionary Algorithms: Multi-Objective Optimization

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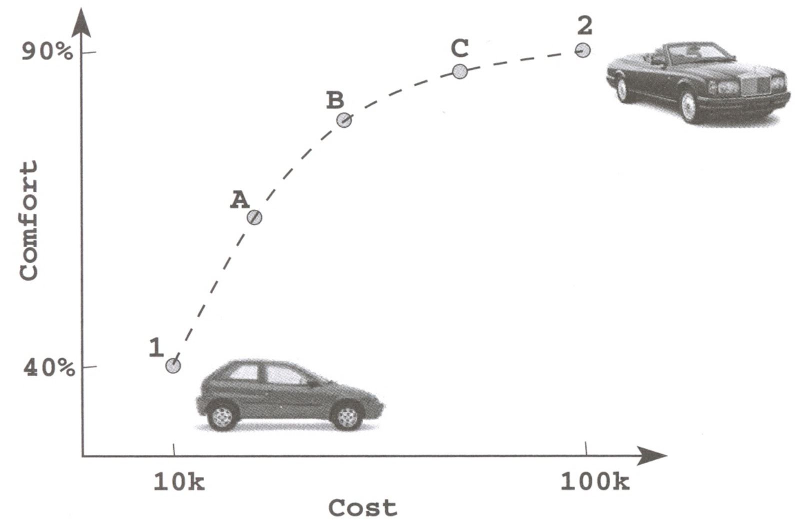
Multi-Objective Optimization

:: Many real-world problems involve multiple objectives

- **Conflicting objectives**
  - A solution that is extreme with respect to one objective requires a compromise in other objectives.
  - A sacrifice in one objective is related to the gain in other objective(s).

  Motivation example: Buying a car
  - two extreme hypothetical cars 1 and 2,
  - cars with a trade-off between cost and comfort – A, B, and C.



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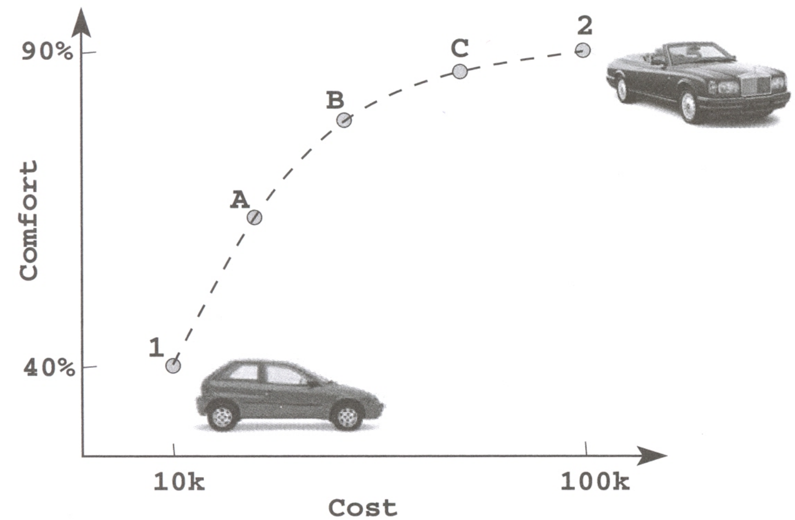

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- **Which solution** out of all of the trade-off solutions **is the best** with respect to all objectives?

  Without any further information those trade-offs are indistinguishable.

  $\Longrightarrow$ **a number of optimal solutions is sought in multiobjective optimization!**

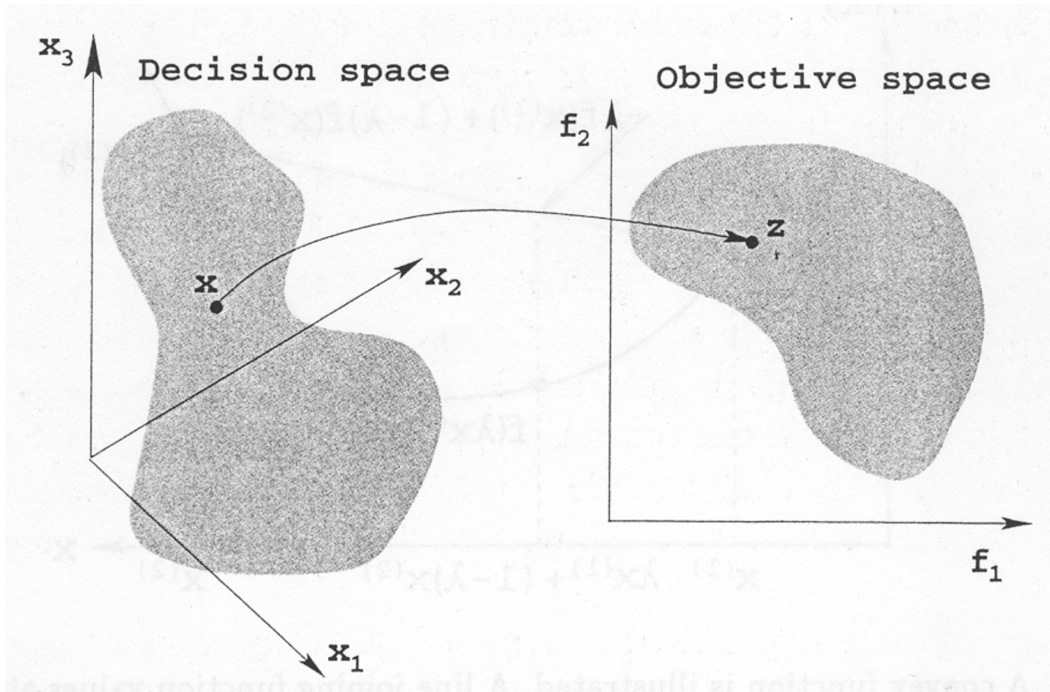
Multi-Objective Optimization: Definition

:: **General form of multi-objective optimization problem**

$$\begin{aligned} \text{Minimize/maximize} \quad & f_m(x), & m = 1, 2, ..., M; \\ \text{subject to} \quad & g_j(x) \geq 0, & j = 1, 2, ..., J; \\ & h_k(x) = 0, & k = 1, 2, ..., K; \\ & x_i^{(L)} \leq x_i \leq x_i^{(U)}, & i = 1, 2, ..., n. \end{aligned}$$

- x is a vector of n decision variables: $x = (x_1, x_2, ..., x_n)^T$;
- **Decision space** is constituted by variable bounds that restrict each variable x_i to take a value within a lower $x_i^{(L)}$ and an upper $x_i^{(U)}$ bound;
- Inequality and equality constraints
- A solution x that satisfies all constraints and variable bounds is a **feasible solution**, otherwise it si called an **infeasible solution**;
- **Feasible space** is a set of all feasible solutions;
- Objective functions $f(x) = (f_1(x), f_2(x), ..., f_M(x))^T$ constitute a multi-dimensional **objective space**.

Decision and Objective Space



- For each solution x in the decision space, there exists a point in the objective space

$$f(x) = z = (z_1, z_2, ..., z_M)^T$$

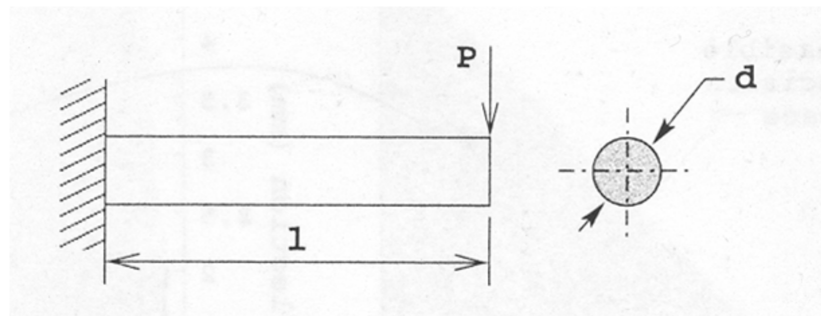
Motivation Example: Cantilever Design Problem

Task is to design a beam, defined by two decision variables

- diameter d,
- length l.

that can carry an end load P and is optimal with respect to the following **objectives**

- f_1 – minimization of $weight$,
- f_2 – minimization of $deflection$.
  Obviously, conflicting objectives!



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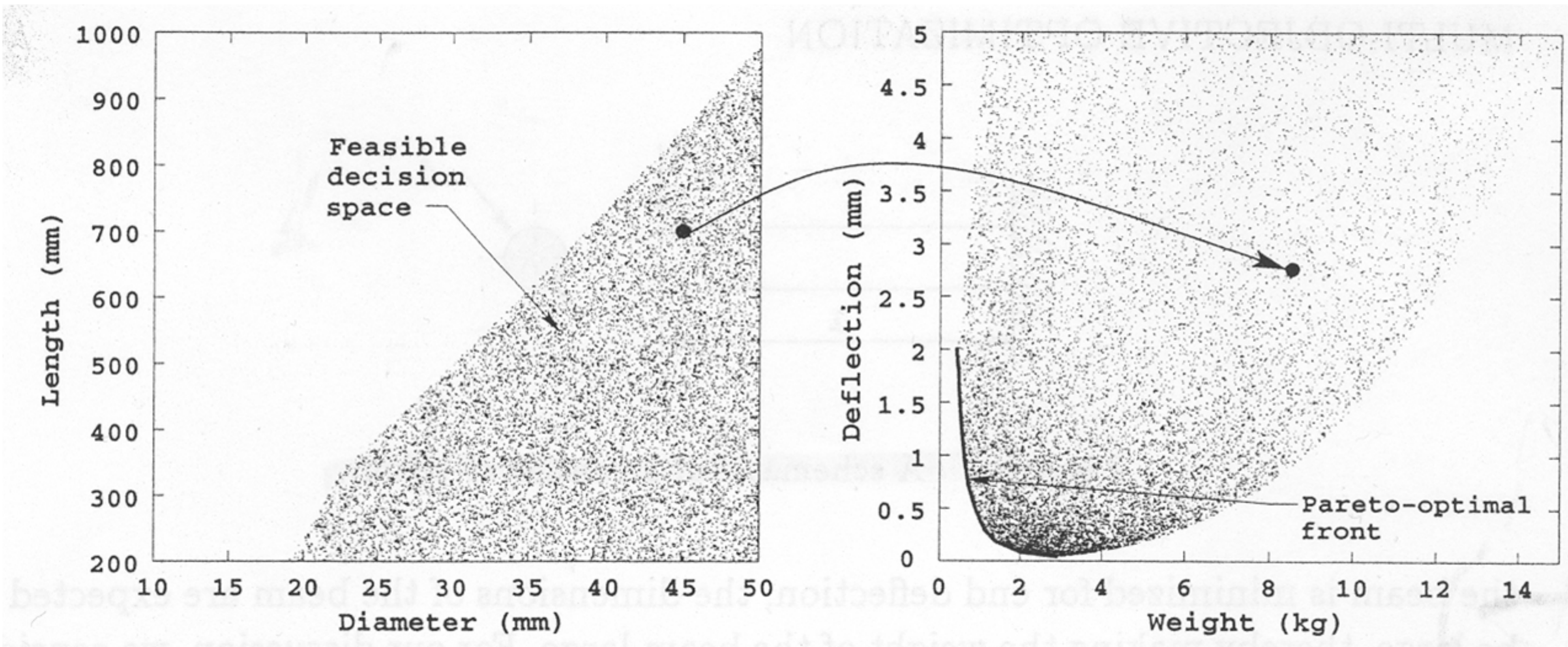
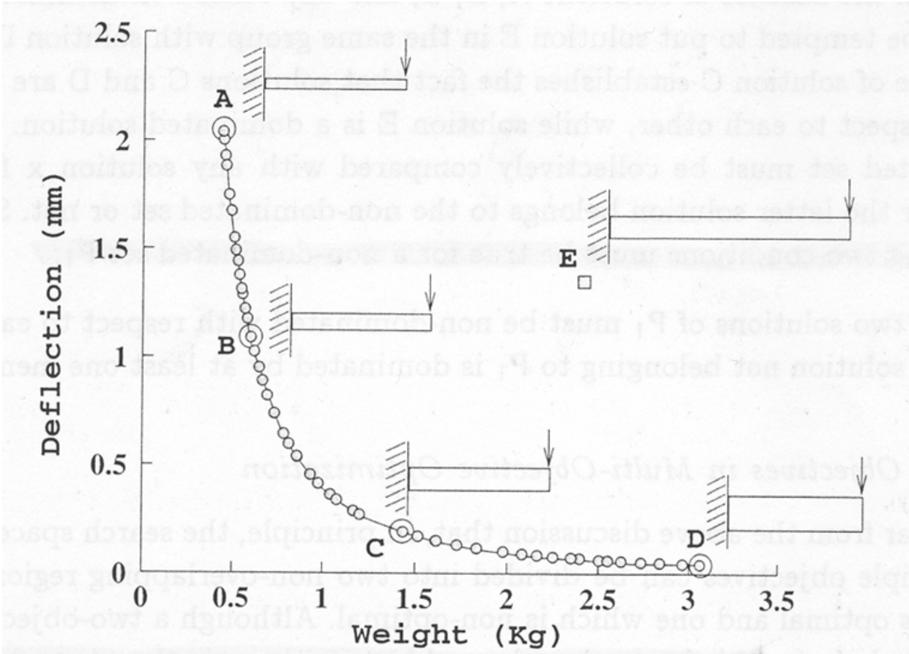
subject to the following **constraints**

- the developed maximum stress σ_{max} is less than the allowable strength S_y,
- the end deflection δ is smaller than a specified limit δ_{max}.

Cantilever Design Problem: Decision and Objective Space



Dominance and Pareto-Optimal Solutions



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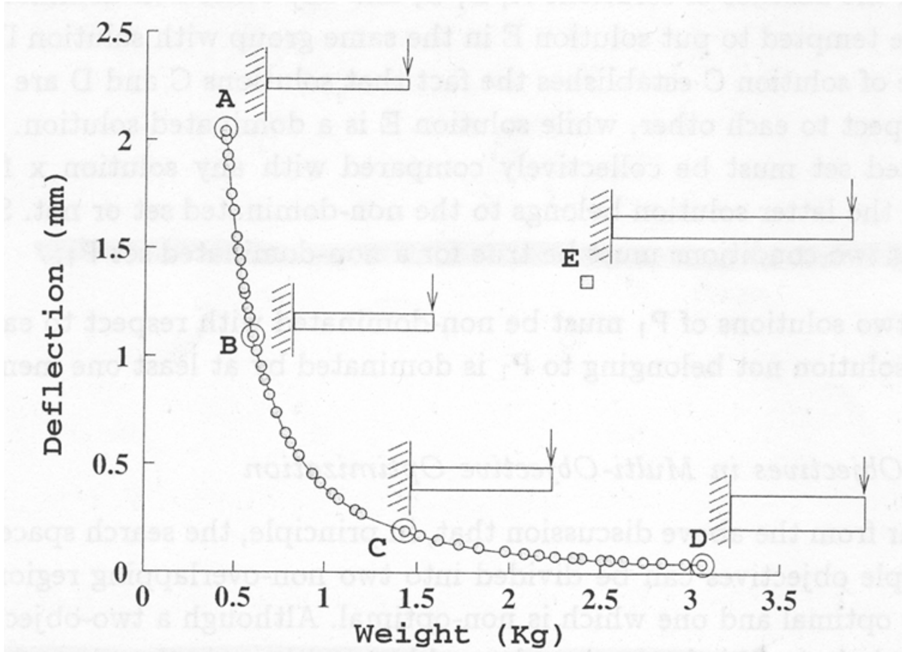

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:: **Domination**: *A solution $x^{(1)}$ is said to dominate the other solution $x^{(2)}$, $x^{(1)} \preceq x^{(2)}$, if $x^{(1)}$ is no worse than $x^{(2)}$ in all objectives and $x^{(1)}$ is strictly better than $x^{(2)}$ in at least one objective.*

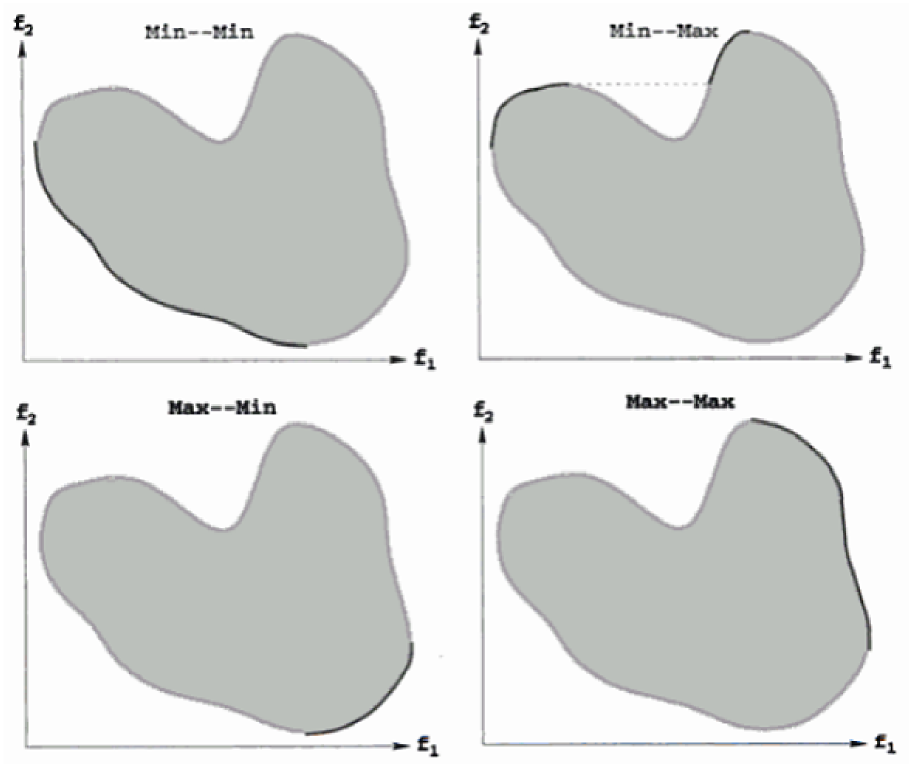
Solutions A, B, C, D are *non-dominated* solutions (Pareto-optimal solutions)

Solution E is *dominated* by C and B (E is non-optimal).

Properties of Dominance-Based Multi-Objective Optimization

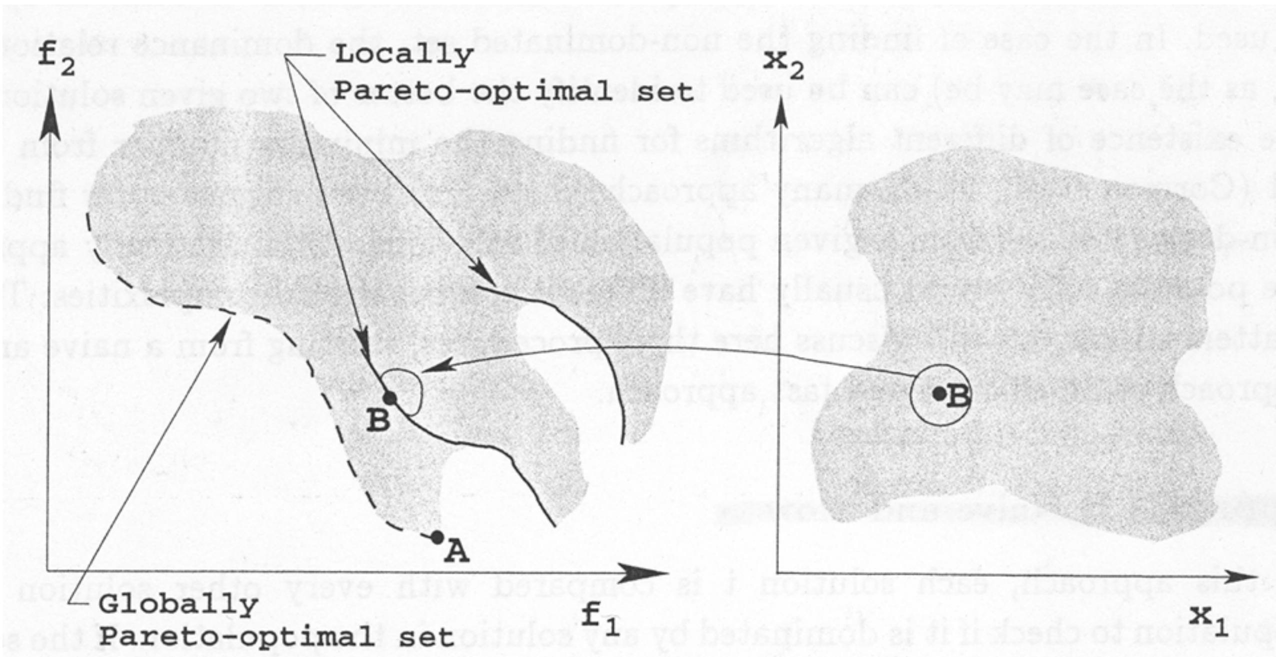
:: **Non-dominated set** – Among a set of solutions P, the noon-dominated set of solutions P' are those that are not dominated by any member of the set P.

The non-dominated set of the entire feasible search space is the **globally Pareto-optimal set**.



Properties of Dominance-Based Multi-Objective Optimization

If for every member x in a set P there exists no solution y (in the neighborhood of x such that $\|y - x\| \leq \epsilon$, where ϵ is a small positive number) dominating any member of the set P, then solutions belonging to the set P constitute a **locally Pareto-optimal set**.



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Goals of Dominance-Based Multi-Objective Optimization

:: Every finite set of solutions P can be divided into two non-overlapping sets

- non-dominated set P_1 – contains all solutions that do not dominate each other, and
- dominated set P_2 – at least one solution in P_1 dominates any solution in P_2.

:: In the absence of other factors (e.g. preference for certain objectives, or for a particular region of the tradeoff surface) there are **two goals of the multi-objective optimization**

- **Quality** – To find a set of solutions as close as possible to the Pareto-optimal front.
- **Spread** – To find a set of solutions as diverse as possible.

Non-Conflicting Objectives

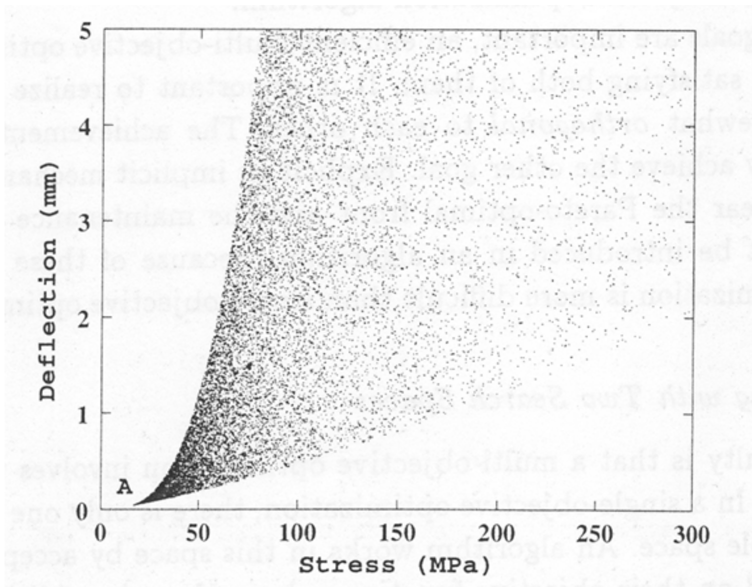
:: There exist multiple Pareto-optimal solutions in a problem only if the objectives are conflicting to each other.

- If this does not hold then the cardinality of the Pareto-optimal set is one.

  This means that the optimum solution corresponding to any objective is the same.

Example: Cantilever beam design problem

- f_1 – minimizing the end deflection δ,
- f_2 – minimizing the maximum developed stress in the beam σ_{max}.



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Differences with Single-Objective Optimization

:: **Two (orthogonal) goals instead of one**

- progressing towards the Pareto-optimal front,
- maintaining a diverse set of solutions in the non-dominated set.

:: **Dealing with two search spaces**

- objective vs. decision space,
- in which space the diversity must be achieved?

:: **No artificial fix-ups**

- weighted sum approach – multiple objectives are weighted and summed together to create a composite objective function.

  Its performance depends on the chosen weights.
- ε-constraint method – chooses one of the objective functions and treats of the objectives as constraints by limiting each of them within certain predefined limits.

  Also depends on the chosen constraint limits.

Difficulties with Classical Optimization Algorithms

- The convergence to an optimal solution depends on the chosen initial solution.
- Most algorithms tend to get stuck to a suboptimal solution.
- An algorithm efficient in solving one optimization problem may not be efficient in solving a different opt. problem.
- Algorithms are not efficient in handling problems having a discrete search space.
- Algorithms cannot be efficiently used on a parallel machine

Multi-Objective Evolutionary Algorithms

- **Pareto Archived Evolution Strategy** (PAES)

  Knowles, J.D., Corne, D.W. (2000) Approximating the nondominated front using the Pareto archived evolution strategy. Evolutionary Computation, 8(2), pp. 149-172

- **Multiple Objective Genetic Algorithm** (MOGA)

  Carlos M. Fonseca, Peter J. Fleming: Genetic Algorithms for Multiobjective Optimization: Formulation, Discussion and Generalization, In Genetic Algorithms: Proceedings of the Fifth International Conference, 1993

- **Niched-Pareto Genetic Algorithm** (NPGA)

  Jeffrey Horn, Nicholas Nafpliotis, David E. Goldberg: A Niched Pareto Genetic Algorithm for Multiobjective Optimization, Proceedings of the First IEEE Conference on Evolutionary Computation, IEEE World Congress on Computational Intelligence, 1994

- **SPEA2**

  Zitzler, E., Laumanns, M., Thiele, L.: SPEA2: Improving the Strength Pareto Evolutionary Algorithm For Multiobjective Optimization, In: Evolutionary Methods for Design, Optimisation, and Control, Barcelona, Spain, pp. 19-26, 2002

- **NSGA**

  Srinivas, N., and Deb, K.: Multi-objective function optimization using non-dominated sorting genetic algorithms, Evolutionary Computation Journal 2(3), pp. 221-248, 1994

- **NSGA-II**

  Kalyanmoy Deb, Samir Agrawal, Amrit Pratap, and T Meyarivan: A Fast Elitist Non-Dominated Sorting Genetic Algorithm for Multi-Objective Optimization: NSGA-II, In Proceedings of the Parallel Problem Solving from Nature VI Conference, 2000

- ...

Non-Dominated Sorting Genetic Algorithm (NSGA)

:: **Common features** with the standard GA

- variation operators – crossover and mutation,
- selection method – Stochastic Reminder Roulette-Wheel,
- standard generational evolutionary model.

:: **What distinguishes NSGA from the SGA**

- fitness assignment scheme which prefers non-dominated solutions, and
- fitness sharing strategy which preserves diversity among solutions of each non-dominated front.

:: **Algorithm NSGA**

1. Initialize population of solutions
2. Repeat
   - Calculate objective values and assign fitness values
   - Generate new population

   Until stopping condition is fulfilled

Fitness Sharing

:: **Diversity preservation method** originally proposed for solving multi-modal optimization problems so that GA is able to sample each optimum with the same number of solutions.

:: **Idea** – diversity in the population is preserved by degrading the fitness of similar solutions

:: **Algorithm** for calculating the shared fitness function value of i-th individual in population of size N

1. calculate *sharing function* value with all solutions in the population according to

$$Sh(d) = \begin{cases} 1 - (\frac{d}{\sigma_{share}})^{\alpha}, & \text{if } d \leq \sigma_{share} \\ 0, & \text{otherwise.} \end{cases}$$

2. calculate niche count nc_i as follows

$$nc_i = \sum_{j=1}^{N} Sh(d_{ij})$$

3. calculate *shared fitness* as

$$f'_i = f_i / nc_i$$

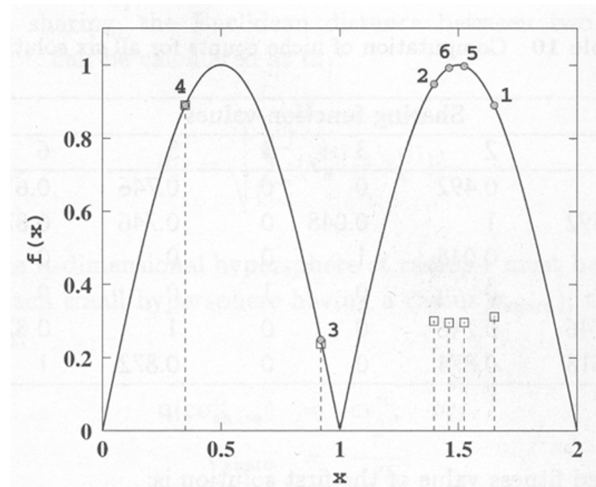
:: **Remark:** If $d = 0$ then $Sh(d) = 1$ meaning that two solutions are identical. If $d \geq \sigma_{share}$ then $Sh(d) = 0$ meaning that two solutions do not have any sharing effect on each other.

Fitness Sharing: Example

:: Bimodal function - six solutions and corresponding shared fitness functions

- $\sigma_{share} = 0.5$, $\alpha = 1$.

Sol. i	String	Decoded value	$x^{(i)}$	f_i	nc_i	f'_i
1	110100	52	1.651	0.890	2.856	0.312
2	101100	44	1.397	0.948	3.160	0.300
3	011101	29	0.921	0.246	1.048	0.235
4	001011	11	0.349	0.890	1.000	0.890
5	110000	48	1.524	0.997	3.364	0.296
6	101110	46	1.460	0.992	3.364	0.295



:: Let's take the first solution

- $d_{11} = 0.0$, $d_{12} = 0.254$, $d_{13} = 0.731$, $d_{14} = 1.302$, $d_{15} = 0.127$, $d_{16} = 0.191$
- $Sh(d_{11}) = 1$, $Sh(d_{12}) = 0.492$, $Sh(d_{13}) = 0$, $Sh(d_{14}) = 0$, $Sh(d_{15}) = 0.746$, $Sh(d_{16}) = 0.618$.
- $nc_1 = 1 + 0.492 + 0 + 0 + 0.746 + 0.618 = 2.856$
- $f'(1) = f(1)/nc_1 = 0.890/2.856 = 0.312$

NSGA: Fitness Assignment

:: **Input**: Set P of solutions with assigned objective values.

:: **Output**: Set of solutions with assigned fitness values (the bigger the better).

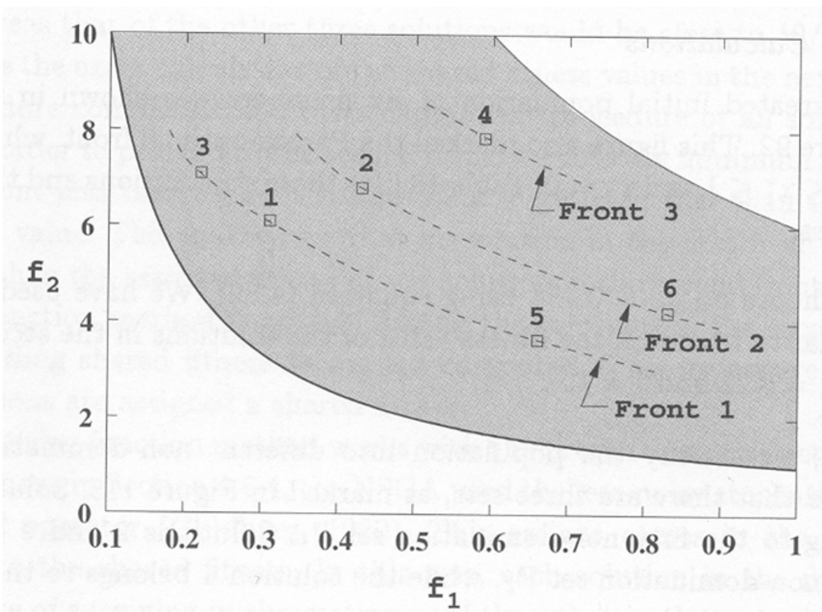
1. Choose sharing parameter σ_{share}, small positive number ϵ,
   initialize $F_{max} = PopSize$ and front counter $front = 1$
2. Find set $P' \subset P$ of non-dominated solutions
3. For each $q \in P'$
   - assign fitness $f(q) = f_{max}$,
   - calculate sharing function with all solutions in P' niche count nc_q among solutions of P' only,
     the normalized Euclidean distance d_{ij} is calculated
   - calculate shared fitness $f'(q) = f(q)/nc_q$.
4. $f_{max} = min(f'(q) : q \in P') - \epsilon$
   $P = P \setminus P'$
   $front = front + 1$
5. If not all solutions are assessed go to step 2, otherwise stop.

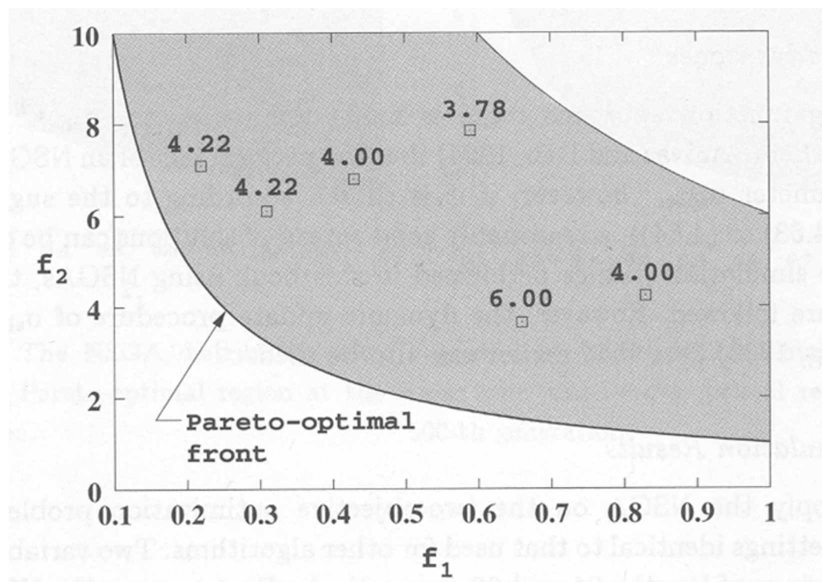
$$d_{ij} = \sqrt{\sum_{k=1}^{n} \left(\frac{x_k^{(i)} - x_k^{(j)}}{x_k^{max} - x_k^{min}}\right)^2}$$

NSGA: Fitness Assignment cont.

:: Example:

- First, 10 solutions are classified into different non-dominated fronts.
- Then, the fitness values are calculated according to the fitness sharing method.
  - The sharing function method is used front-wise.
  - Within a front, less dense solutions have better fitness values.





NSGA: Conclusions

:: Computational complexity

- Governed by the non-dominated sorting procedure and the sharing function implementation.
  - **non-dominated sorting** – complexity of $O(MN^3)$.
  - **sharing function** – requires every solution in a front to be compared with every other solution in the same front, total of $\sum_{j=1}^{\rho} |P_j|^2$, where ρ is a number of fronts.
    Each distance computation requires evaluation of n differences between parameter values.
    In the worst case, when $\rho = 1$, the overall complexity is of $O(nN^2)$.

:: Advantages

- Assignment of fitness according to non-dominated sets – makes the algorithm converge towards the Pareto-optimal region.
- Sharing allows phenotypically diverse solutions to emerge.

:: Disdvantages

- sensitive to the sharing method parameter σ_{share}.
  - some guidelines for setting the parameter based on the expected number of optima q.

$$\sigma_{share} = \frac{0.5}{\sqrt[n]{q}}$$

  - or dynamic update procedure of σ_{share}.

NSGA-II

:: **Fast non-dominated sorting approach**

- Computational complexity of $O(MN^2)$.

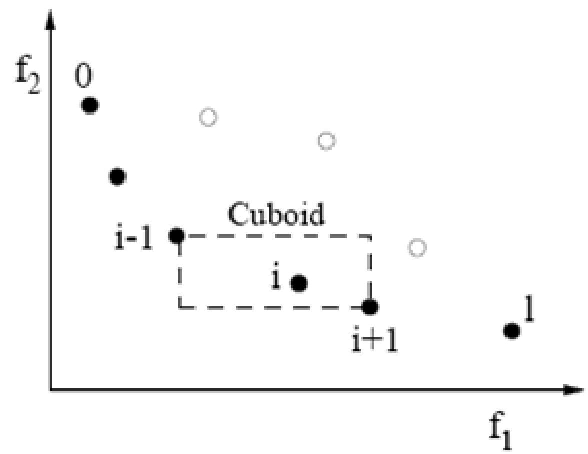
:: **Diversity preservation**

- the sharing function method is replaced with a **crowded comparison approach**,
- parameterless approach.

:: **Elitist evolutionary model**

NSGA-II: Diversity preservation

:: Density estimation



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:: Crowded comparison operator

Every solution in the population has two attributes

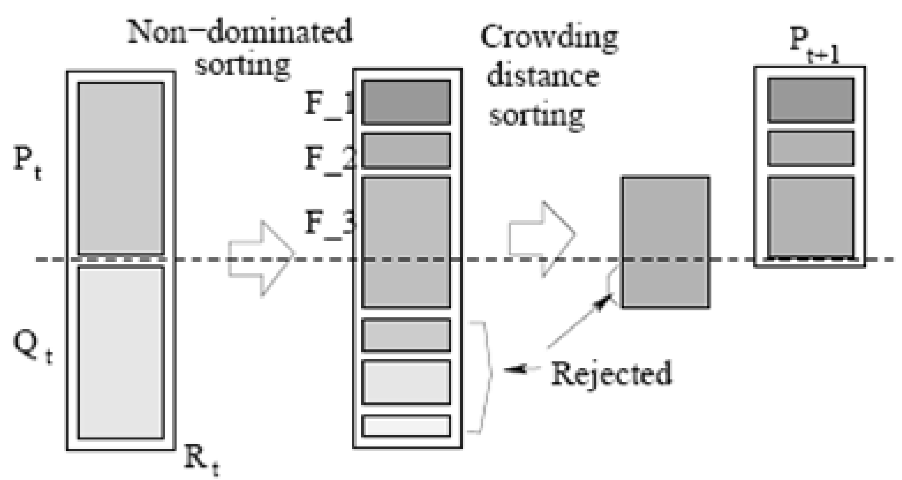
1. non-domination rank (i^{rand}), and
2. crowding distance ($i^{distance}$).

A partial order $\prec_n$ is defined as:

$$i \prec_n j \text{ if}(i^{rank} < j^{rank}) \text{ or } ((i^{rank} = j^{rank}) and (i^{distance} > j^{distance}))$$

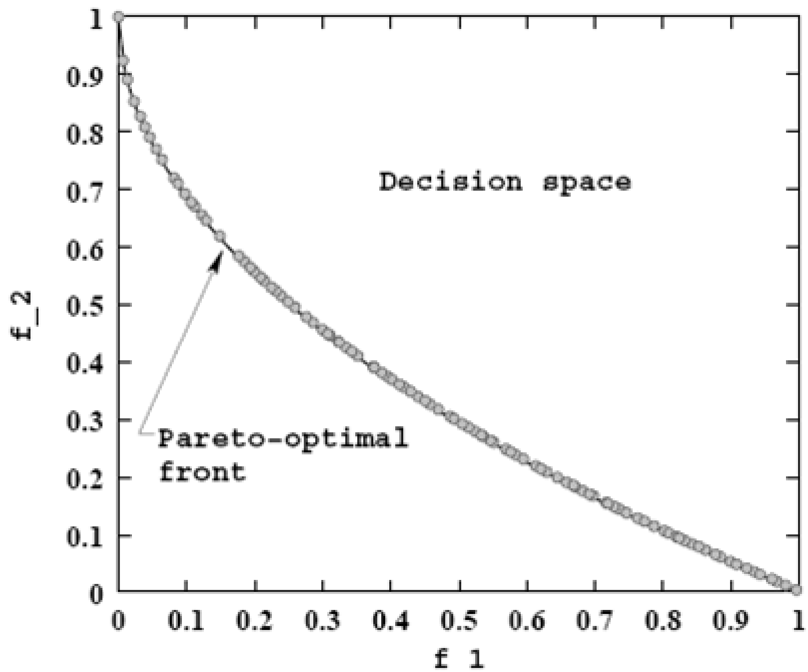
NSGA-II: Evolutionary Model

1. Current population P_t is sorted based on the non-domination
   Each solution is assigned a fitness equal to its non-domination level (1 is the best).
2. The usual binary tournament selection, recombination, and mutation are used to create a child population Q_t of size N.
3. Combined population $R_t = P_t \cup Q_t$ is formed.
   Elitism is ensured.
4. Population P_{t+1} is formed according to the following schema



NSGA-II: Simulation Results

Problem with continuous Pareto-optimal front



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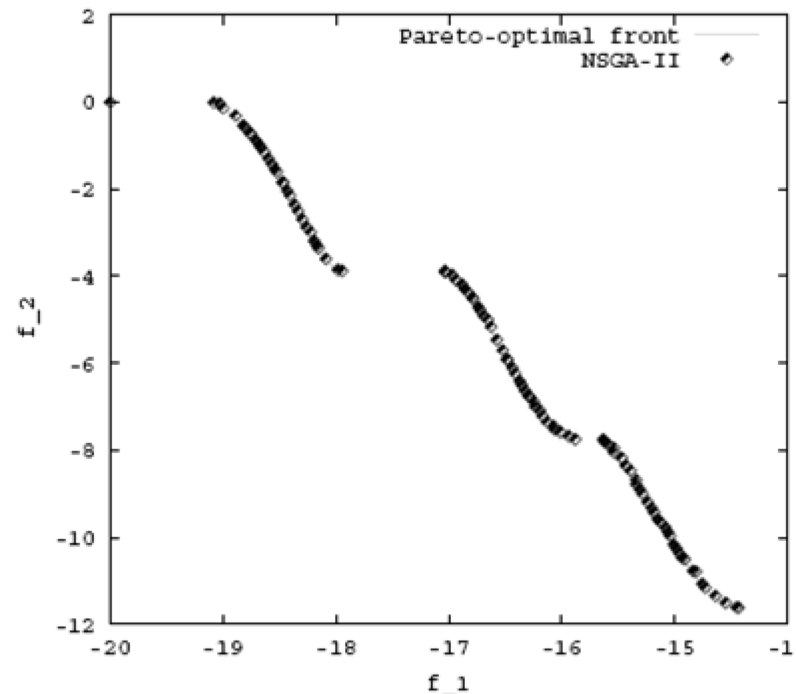

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Multiobjective 0/1 Knapsack Problem

Definition:

Given a set of m items and a set of n knapsacks, with $p_{i,j}$ being profit of item j according to knapsack i, $w_{i,j}$ being weight of item j according to knapsack i, and c_i being capacity of knapsack i, find a vector $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in \{0,1\}^m$, such that

$$\forall i \in \{1, 2, \ldots, n\} : \sum_{j=1}^{m} w_{i,j} \cdot x_j \leq c_i$$

and for which $f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$ is maximum, where

$$f_i(\mathbf{x}) = \sum_{j=1}^{m} p_{i,j} \cdot x_j$$

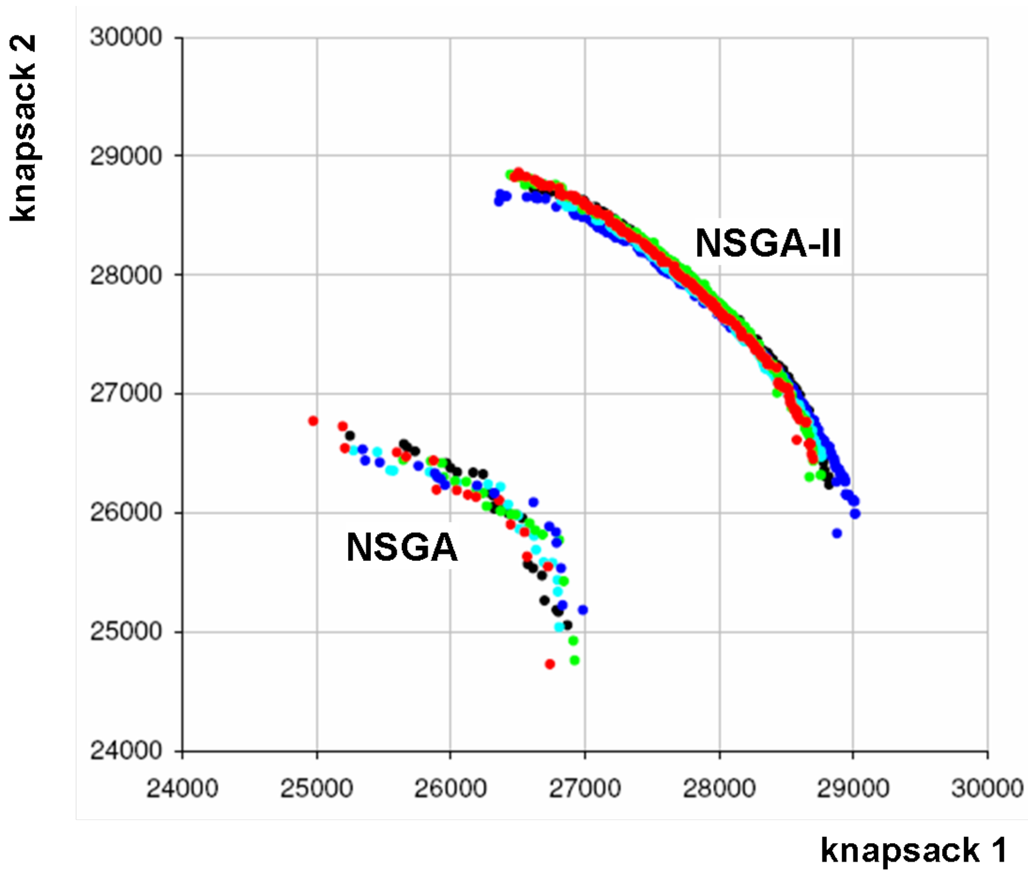
and $x_j = 1$ iff item j is selected.

A well-known **NP hard** combinatorial optimization problem.

Simulation Results: NSGA vs. NSGA-II

Comparison of NSGA nad NSGA-II on bi-objective 0/1 Knapsack Problem with 750 items.

NSGA-II outperforms NSGA with respect to both performance measures.



NSGA-II: Constraint Handling Approach

:: **Binary tournament selection** with modified domination concept is used to choose the better solution out of the two solutions i and j, randomly picked up from the population.

:: In the presence of constraints each solution in the population can be either **feasible** or **infeasible**, so that there are the following three possible situations:

- both solutions are feasible,
- one is feasible and other is not,
- both are infeasible.

:: **Constrained-domination**: A solution i is said to constrained-dominate a solution j, if any of the following conditions is true

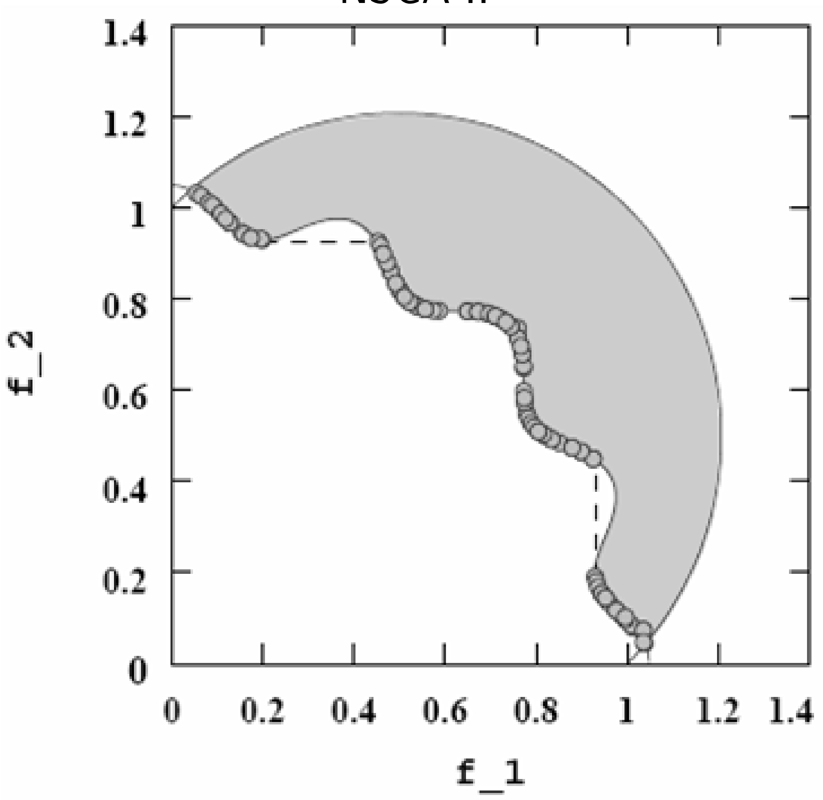
1. Solution i is feasible and solution j is not.
2. Solutions i and j are both infeasible, but solution i has a smaller overall constraint violation.
3. Solutions i and j are feasible, and solution i dominates solution j.

NSGA-II: Simulation Results cont.

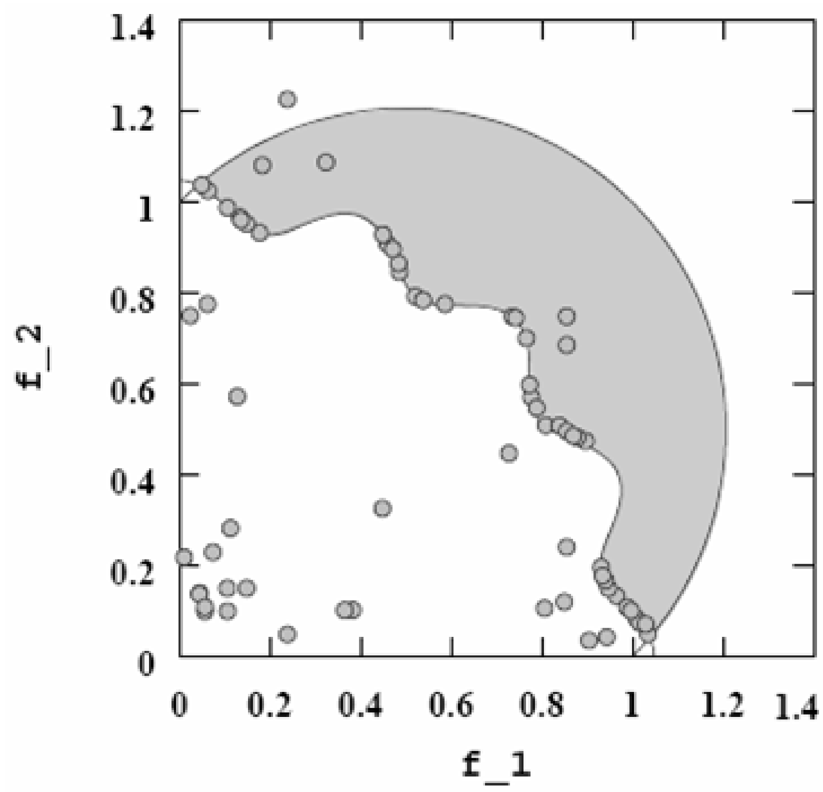
Comparison of NSGA-II and Ray-Kang-Chye's Constraint handling approach

- Ray, T., Tai, K. and Seow, K.C. [2001] "Multiobjective Design Optimization by an Evolutionary Algorithm", Engineering Optimization, Vol.33, No.4, pp.399-424

NSGA-II



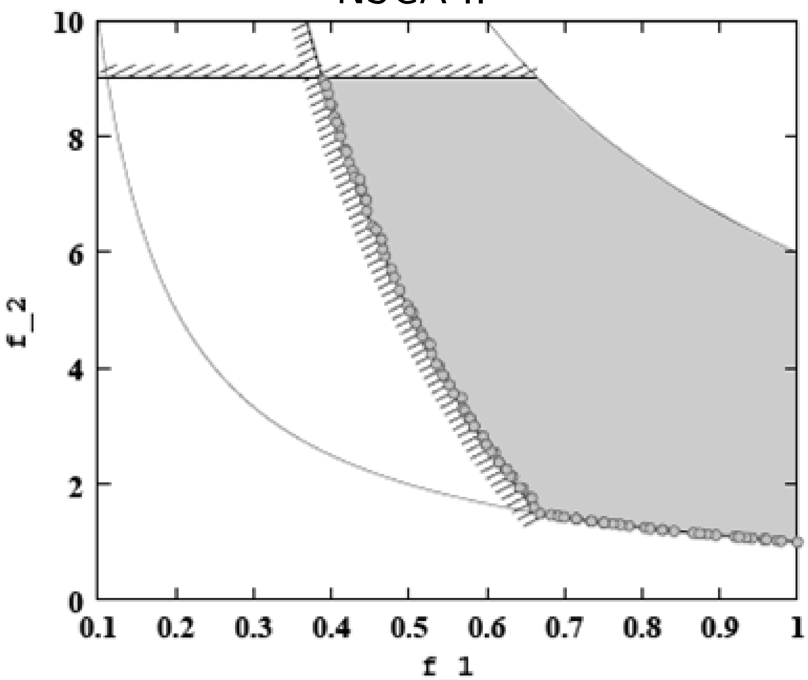
Ray-Kang-Chye's



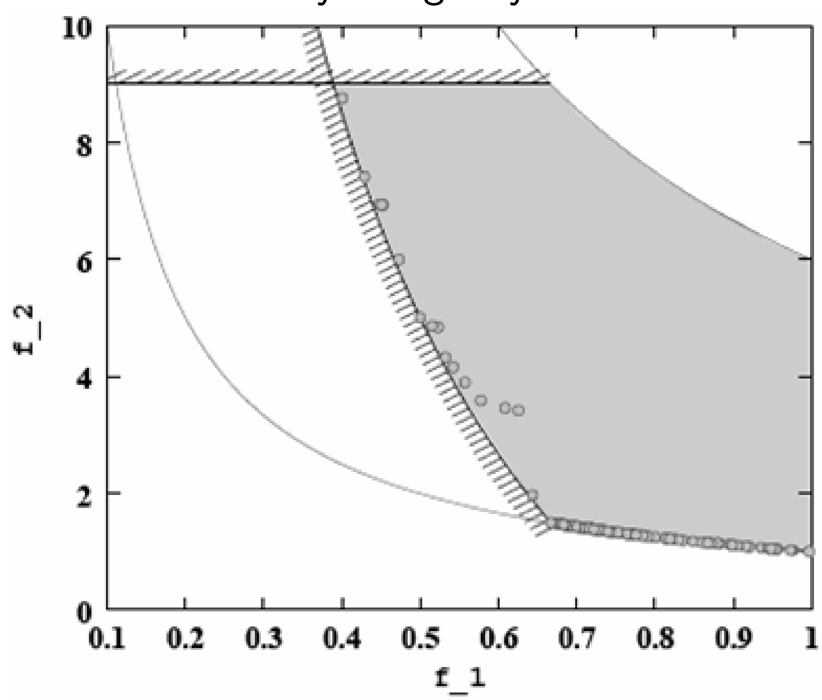
NSGA-II: Simulation Results cont.

Comparison of NSGA-II and Ray-Kang-Chye's Constraint handling approach

NSGA-II



Ray-Kang-Chye's



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MOEA Performance Measures

:: The result of a MOEA run is not a single scalar value, but a collection of vectors forming a non-dominated set.

1. Comparing two MOEA algorithms requires comparing the non-dominated sets they produce. However, there is no straightforward way to compare different non-dominated sets.

:: Three goals that can be identified and measured:

1. The distance of the resulting non dominated set to the Pareto-optimal front should be minimized.
2. A good (in most cases uniform) distribution of the solutions found is desirable.
3. The extent of the obtained non dominated front should be maximized, i.e., for each objective, a wide range of values should be present.

MOEA Performance Measures cnd.

:: Properties of metrics for comparing non-dominated sets

1. Pareto compatibility – a comparison metric R is compatible with an outperformance relation if for each pair of non-dominated sets A nad B, such that $A \leq B$, R will evaluate A as being better than B.

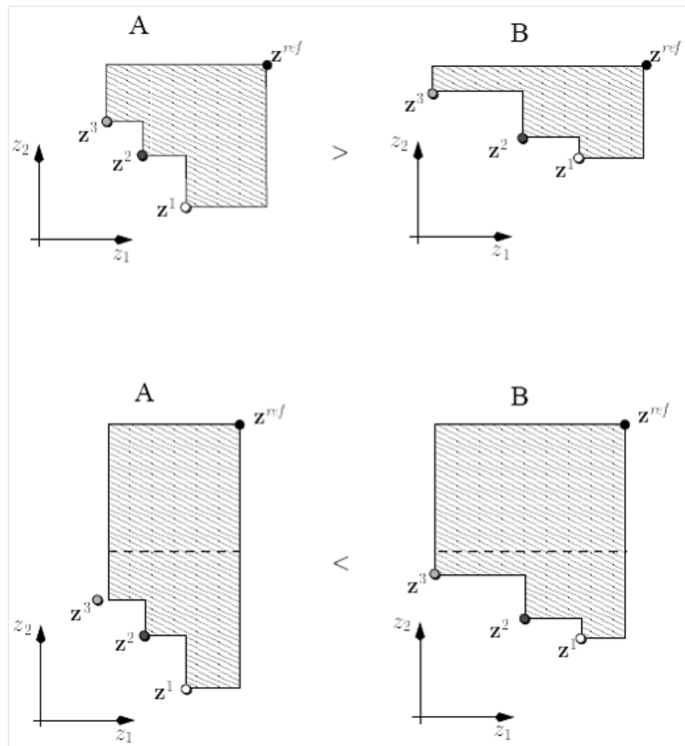
   Outperformance relation $\leq$ – a non-dominated set A completely outperforms set B if each point in B is dominated by a point in A.

2. Direct comparative metric – compares A and B directly using a scalar measure $R(A, B)$ to describe how much better A is than B.
3. Reference metric – use a reference set; it scores both sets against this reference set and compares the results.
4. Independent metric – measures some property of each set that is not dependent on any other, or any reference set.
5. Transitive metric – induces a complete ordering of all possible non-dominated sets. It ensures that if A beats B, and B beats C then it is always true that A beats C.
6. Cardinal metrics – counts the number of vectors in sets.

S Metric

Size of the space covered $S(X)$ – it calculates the *hypervolume* of the multi-dimensional region enclosed by a set A and a *reference point* (usualy so-called *Utopian* point). The hypervolume expresses the size of the region A dominates.

So, the bigger the value of this measure the better the quality of A is, and vice versa.



S Metric cnd.

Pros:

- Compatible with the outperformance relations.
- Independent.
- Differentiates between different degrees of complete outperformance of two sets.
- Scaling independent.
- Intuitive meaning/interpretation.

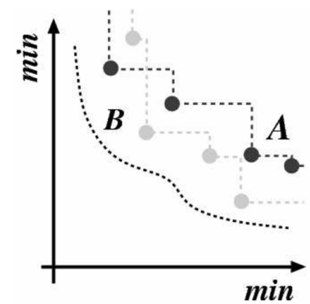
Cons:

- Requires defing some upper boundary of the region.
  This choice does affect the ordering of non-dominated sets.
- It has a large computational overhead, $O(n^{k+1})$, rendering it unusable for many objectives or large sets.
- It multiplies apples by oranges, that is, different objectives together.

C Metric

Coverage of two sets $C(X, Y)$ – given two sets of non-dominated solutions X and Y found by the compared algorithms, the measure $C(X, Y)$ returns a ratio of a number of solutions of Y that are dominated by or equal to any solution of X to the whole set Y.

- It returns values from the interval [0, 1].
- The value $C(X, Y) = 1$ means that all solutions in Y are covered by solutions of the set X. And vice versa, the value $C(X, Y) = 0$ means that none of the solutions in Y are covered by the set X.
- Always both orderings have to be considered, since $C(X, Y)$ is not necessarily equal to $1 - C(Y, X)$.



$C(A, B) = 0.75,$
$C(B, A) = 0.25$

Properties:

- It has low computational overhead.
- The non-symmetric nature of C complicates the analysis of its compatibility with the outperformance relations.
- If two sets are of different cardinality and/or the distributions of the sets are non-uniform, then it gives unreliable results.

C Metric cnd.

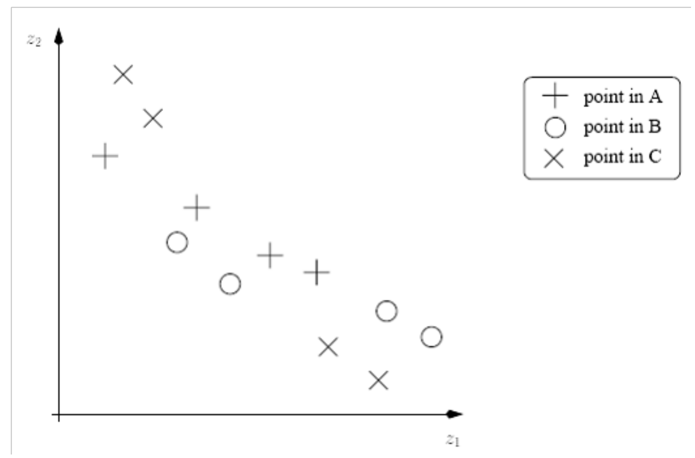
Properties:

- Any pair of C metric scores for a pair of sets A and B in which neither $C(A, B) = 1$ nor $C(B, A) = 1$, indicates that the two sets are incomparable according to the weak outperformance relation.
- It is cycleinducing – if three sets are compared using C, they may not be ordered.

Example:

- $C(A, B) = 0$, $C(B, A) = 3/4$
- $C(B, C) = 0$, $C(C, B) = 1/2$
- $C(A, C) = 1/2$, $C(C, A) = 0$

C considers B better than A, A better than C, but C better than B.



©Knowles J. and Corne D.: On Metrics for Comparing Non-Dominated Sets.

Reading

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