Knowledge in multi-agent systems

Task 5. Who is the best?

Deadline for submition: 9.5. 09:00 am.

Three Masters of Logic want to find out who was the wisest amongst them and asked their Grand Master to resolve their dispute. "Easy," the old sage said:

"I will blindfold you and paint either red, or blue dot on each man's forehead. When I take your blindfolds off, if you see at least one red dot, **raise your hand**. The one, who guesses the color of the dot on his forehead first, wins.,, And so it was done …
When he took their blindfolds off, all three men raised their hands as the rules required, and sat in silence pondering. Finally, one of them said: "I have a red dot on my forehead."

A: How did the winner guess?

Task 5 continued

After losing the "Spot on the Forehead" contest, the two defeated Puzzle Masters complained that the winner had made a slight pause before raising his hand … . And so the Grand Master vowed to set up a ***truly fair test*** to reveal the best logician amongst them.

He showed the three men 5 hats - two white and three black. Then he turned off the lights in the room and put a hat on each Puzzle Master's head. After that the old sage hid the remaining two hats, but before he could turn the lights on, one of the Masters (as chance would have it, the winner of the previous contest) announced the color of his hat. And he was right once again.

B: What color was the winner´s hat? What could have been his reasoning?

Task 6. Card game “Aces and nines”

3 players have a deck consisting of **4 ACEs** a **4 NINEs**. Each gets 2 cards, 2 remaining are left face down. None of the players looks at his/her cards - instead he/she raises them to his/her forehead so that **the others** can see them. All the players take turns trying to determine their own cards. If a player does not know his/her cards he/she must say so. The first, who announces “I know!” is the winner!

Given **4 ACEs + 4 NINEs,** each of the players **1,2,3** can have **NN, NA** or **AA.**

Round a)

1. Both the Player**1** and Player**2** say “I cannot determine my cards.”
2. The Player**3** can see, that **1AA** and **2NN**.
3. What will be the claim of the Player**3**?

Round b)

1. You are the Player**1** and you can see, that there holds **2NN** and **3AN**.
2. In the first turn no one was able to determine what he or she is holding. Now is your turn.
3. What will you announce?

Round c)

1. You are the Player**2** and you can see **1AN** and **3AN**.
2. In the first turn no one was able to determine what he or she is holding.
3. Player**1** cannot determine her cards at her second turn either.
4. What about you at your second turn ?

Example 1: “Task 5” – its language and state space

Having **4 ACEs** and **4 NINEs** each player **1,2** or **3** can hold one of the three possibilities **NN, AN** or **AA.**

Φ = {**1AA, 1AN, 1NN, 2AA, 2AN, 2NN, …**}

S = { (**AA-AA-NN**), (**AA-AN-AN**), (**AA-NN-AA**), …}

π((**AA-AA-NN**))(**2AA** & **3NN**) = *true*

π((**AA-AA-NN**))(**1NN**) = *false* ...

$M = (S, \pi, K_1, K_2, K_3)$

Which formula expresses the fact that the Player2 does not know his cards?

Např. K_2 (**2AA** ∨ **2AN** ∨ **2NN**) & ¬ K_2 **AA** & ¬ K_2 **AN** & ¬ K_2 **NN**

Example 2. Card game for 2 players and 3 cards A,B, C

$G = \{ 1, 2 \}$ players **1** and **2**

c = { A, B, C} three cards **A**, **B**, **C**

Primitive propositions $\mathbf{\Phi} = \{ 1A, 1B, 1C, 2A, 2B, 2C\}$

1A means "Player**1** holds the card **A",** …

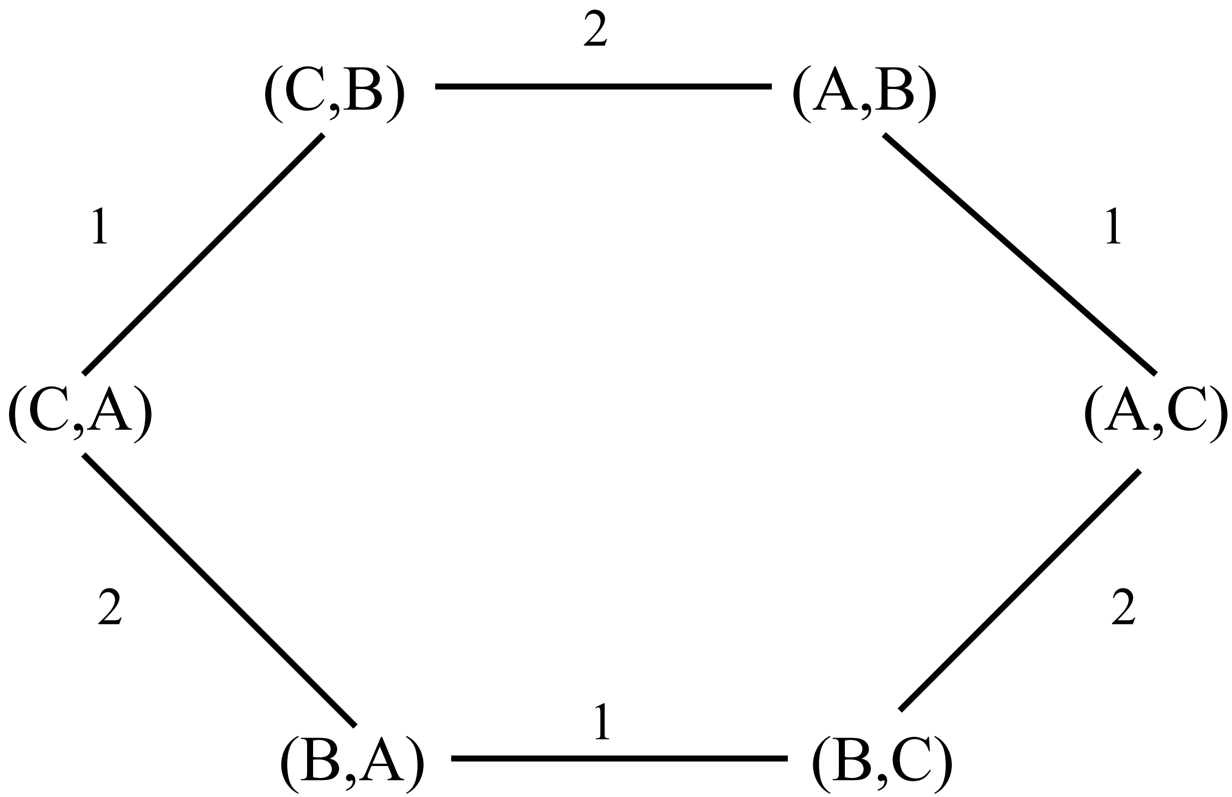
Possible states ***S*** = { (A,B), (A,C), (B, A), (B, C), (C, A), (C, B) }

(A,B): Player1 holds A and Player2 holds B, ...

$\boldsymbol{\pi}((A, B))(1A) = \textit{true}$ $\boldsymbol{\pi}((A, B))(1B) = \textit{false}$ ...

$\boldsymbol{M} = (\mathbf{S}, \boldsymbol{\pi}, K_1, K_2)$

Let us denote as *M* the Kripke structure given by this graph:



$$K_1 = \{[(A,B),(A,C)],[(B,A),(B,C)],[(C,B),(C,A)]\}$$

$$K_2 = \{[(C,A),(B,A)],[(A,B),(C,B)],[(A,C),(B,C)]\}$$

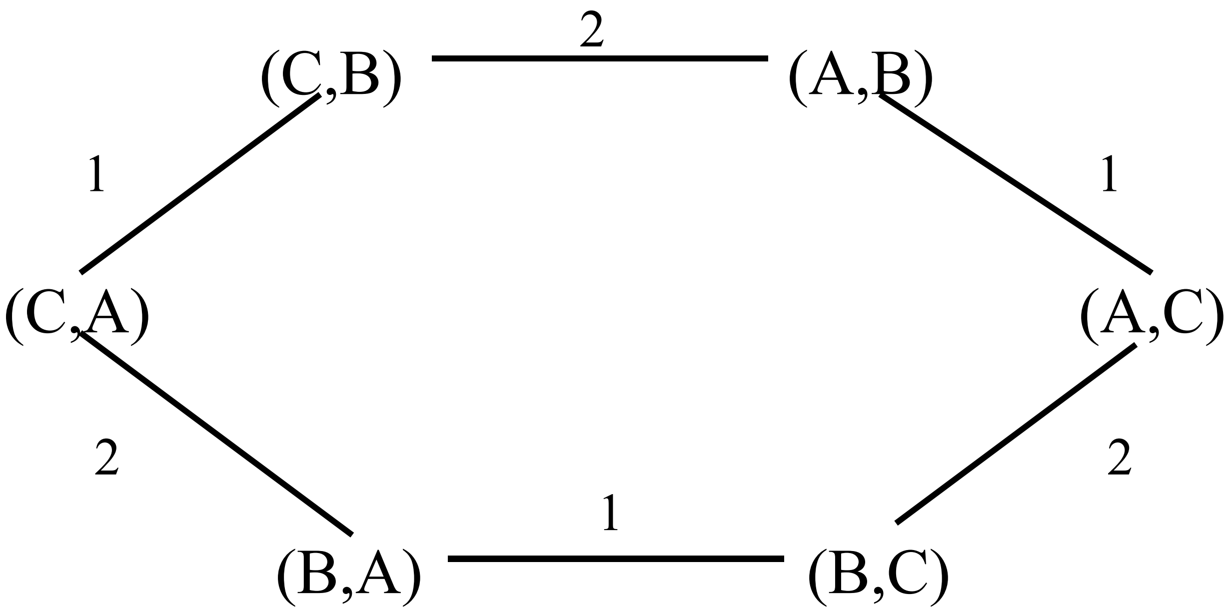
Which card a player can see: his/her own, that of the partner or both?

This example points to the fact, that the Kripke structure has to include even states the agent does not consider as possible.

For example in the state (***A,B***) the *Player1* knows, that the state (***B,C***) is not possible. (*Player1* knows the card it holds, namely the card *A*.)

All over it *Player1* considers it possible, that *Player2* considers the state (***B,C***) as one of the alternative possibilities – it has to be included in the Kripkeho structure. How is this depicted in the graph? There is no edge labeled by ***1*** from (***A,B***) to (***B,C***).

> There is an edge labeled by 1 from (***A,B***) to (***A,C***), and an edge labeled by 2 from (***A,C***) to (***B,C***).



It is easy to verify in ***M*** that

$$(M,(A,B)) \models K_1(2B \vee 2C)$$

$$(M,(B,C)) \models K_2(2C) \wedge K_2(1A \vee 1B)$$

Can we verify more complex claims?

Let $\boldsymbol{M}$ = (S, π,.. K_1 , K_2 , K_3 ,..., K_n) be a Kripke structure *Let* $\boldsymbol{s} \in$ S be any of ***M***'s states and let ***A***, ***B*** **be** any formulas of the considered language.

Verify, that **there must hold**

i. $(\boldsymbol{M}, \boldsymbol{s}) \models (K_i \mathbf{A} \;\&\; K_i (\boldsymbol{A} \rightarrow \boldsymbol{B})) \rightarrow K_i \boldsymbol{B}$

ii. If any K_i is reflexive, then $(\boldsymbol{M}, \boldsymbol{s}) \models K_i \boldsymbol{A} \rightarrow \boldsymbol{A}$

iii. If any K_i is transitive then $(\boldsymbol{M}, \boldsymbol{s}) \models K_i \boldsymbol{A} \rightarrow K_i K_i \boldsymbol{A}$

iv. If any K_i is symetric and transitive, then

$$(\boldsymbol{M}, \boldsymbol{s}) \models \neg K_i \boldsymbol{A} \rightarrow K_i(\neg K_i \boldsymbol{A})$$

Let ***P*** be a Kripke structure with possibility relations all of which are equivalences. In any state of ***P*** all the formulas **i.** to **iv.** are valid !

Let us define

$$(M,s) \models E_G A \quad \Leftrightarrow \quad (M,s) \models K_i A \; for\ all \;\; i \in G$$

$$(M,s) \models C_G A \quad \Leftrightarrow \quad (M,s) \models E_G^k A \; for\ all \; 1 \leq k$$

Both notions have an interesting graphical interpretation:

Let ***G*** be a nonempty set of agents. We say that the state ***t*** is ***G*-reachable** from the state ***s*** in $0 < k$ steps, if there is a sequence of states

$$s \equiv s_0, s_1, \ldots, s_k \equiv t$$

Such that, for any $j, 0 \leq j < k$ there exists $i \in G$ such that

$$(s_j, s_{j+1}) \in K_i.$$

We say that ***t*** is **G**-reachable from ***s***, if ***t*** is ***G***-reachable in finite number of steps.

Lemma.

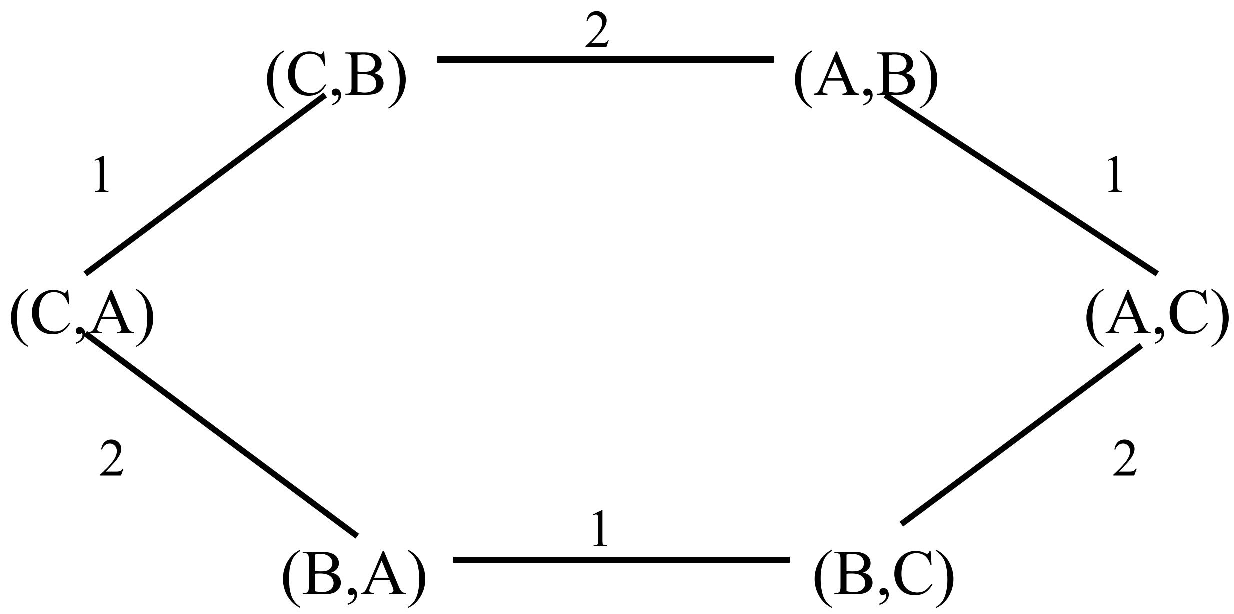
$$(i)\ (M,s) \models E_G^k A \quad \Leftrightarrow \quad (M,t) \models A \quad \textit{for any } t, \; G-\textit{reachable in } k \textit{ steps}$$

$$(ii)\ (M,s) \models C_G A \quad \Leftrightarrow \quad (M,t) \models A \quad \textit{for any } t, \; G-\textit{reachable from } s.$$

Proof.

(i) By induction on k, (ii) is a consequence of (i).

Both claims are valid for any admissibility relations K_i (Here, there is no need to limit our attention to equivalence relations, because the proof does not require anything special from admissibility relations).

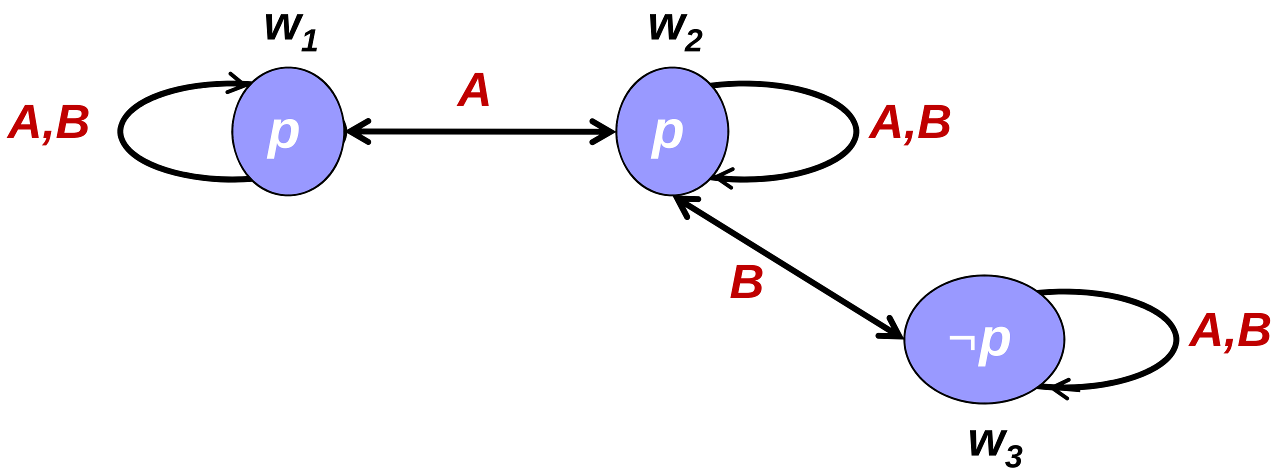


Try to verify the following claims using the reachability property!

$$(M, (A, B)) \models C_G(1A \vee 1B \vee 1C)$$

$$(M, (A, B)) \models C_G(1B \rightarrow (2A \vee 2C))$$

Is there a difference between common knowledge and a fact everyone knows?



Compare truth evaluation of formula $\boldsymbol{E_{\{A,B\}}\, p}$ *and* $\boldsymbol{C_{\{A,B\}}\, p}$ *in the state* $\boldsymbol{w_1}$ *!*

Following formulas are valid:

$$(i) \quad (C_G A \wedge C_G (A \rightarrow B)) \rightarrow C_G B$$

$$(ii) \quad C_G A \rightarrow A$$

$$(iii) \quad C_G A \rightarrow C_G C_G A$$

$$(iv) \quad \neg C_G A \rightarrow C_G \neg C_G A$$

The assumptions about the relations K_i *are the same as in case of knowledge of individual agents.*