Short introduction to motion control of manipulators

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Preliminaries

- Many different robots with various complexity of dynamics and dimensionality of state-action space:
  - triple pendulum,
  - two-arm manipulator,
  - mobile platform with auxiliary articulated sub-tracks and lockable differential.
- Usually given
  - **states**: $\mathbf{x} \in X$,
  - **actions**: $\mathbf{u} \in U$,
  - sometimes **motion model**: $\mathbf{x}' = f(\mathbf{x}, \mathbf{u}) : X \times U \to X$

Problem definition

- **Hard problem:** Find the optimal collision-free trajectory from start state(s) $\mathbf{x}_s$ to goal state(s) $\mathbf{x}_g$ which respect dynamic constraints of the robot. (e.g. LQR-trees or Guided Policy Search)

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- **If robot's dynamic is weak:** one possible approximation to the hard problem is to solve the easier problem and interpolate the path by a smooth trajectory.

Outline

First part of this lecture: solve the easier problem and replace the path by a smooth curve.

1. Short summary of path planning via Rapidly Exploring Random Trees (RRT).
2. Introduce cubic spline interpolation of the planned path.

Second part of this lecture: what to do if motion model is not available?

3. Short intro to reinforcement learning for robotics.

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- Search in continuous high-dimensional space can get stuck in a local minimum, since you can expand infinite number of nodes in a small region.
- RRT [1] efficiently search non-convex, high-dimensional spaces by randomly building a space-filling tree.
- Initial publication [1] has over 1200 citations.

[1] LaValle, Steven M. (October 1998). "Rapidly-exploring random trees: A new tool for path planning"(PDF). Technical Report (Computer Science Department, Iowa State University) (TR 98-11).

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   - if possible: add node $\mathbf{x}$ and edge $[\mathbf{y}^*, \mathbf{x}]$ to the search tree.
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   - otherwise: throw $\mathbf{x}$ away.
6. Repeat from 2 until a feasible path is found.

Output: a collision-free path if exist (after infinite number of nodes expanded)

Speed-ups of Rapidly Exploring Random Tree

- Bias to goal (greediness), e.g.
  - with $P = 0.95$ choose $\mathbf{x}$ at random from X,
  - with $P = 0.05$ choose $\mathbf{x} := \mathbf{x}_g$.

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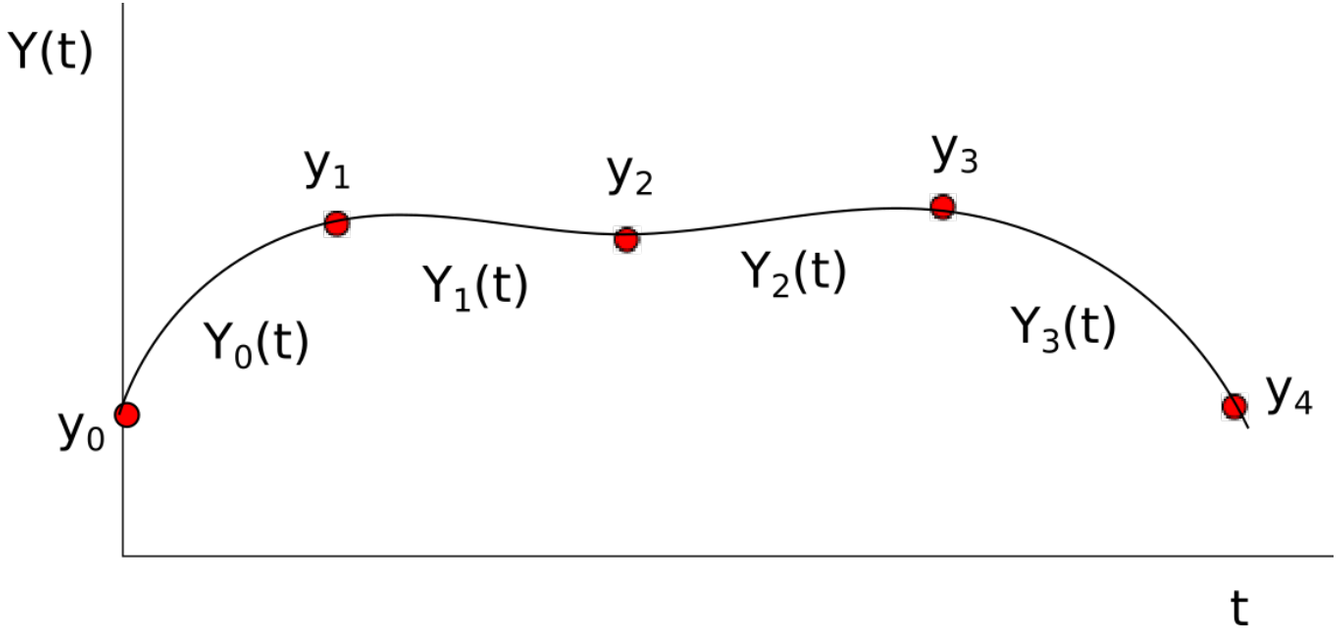
- Probabilistic completeness: if nodes number reaches infinity than you find a feasible path (if exists) with $P = 1$.
- if nodes number reaches infinity then RRT* find the optimal path
- if accurate dynamic motion model is known (i.e. $\dot{\mathbf{x}}$, $\ddot{\mathbf{x}}$ relations are determined), then RRT searches for trajectory in lifted state-space $[\mathbf{x},\ \dot{\mathbf{x}},\ \ddot{\mathbf{x}}]^\top$.
- if only kinematic model is available we have to generate smooth trajectory

Smooth path interpolation



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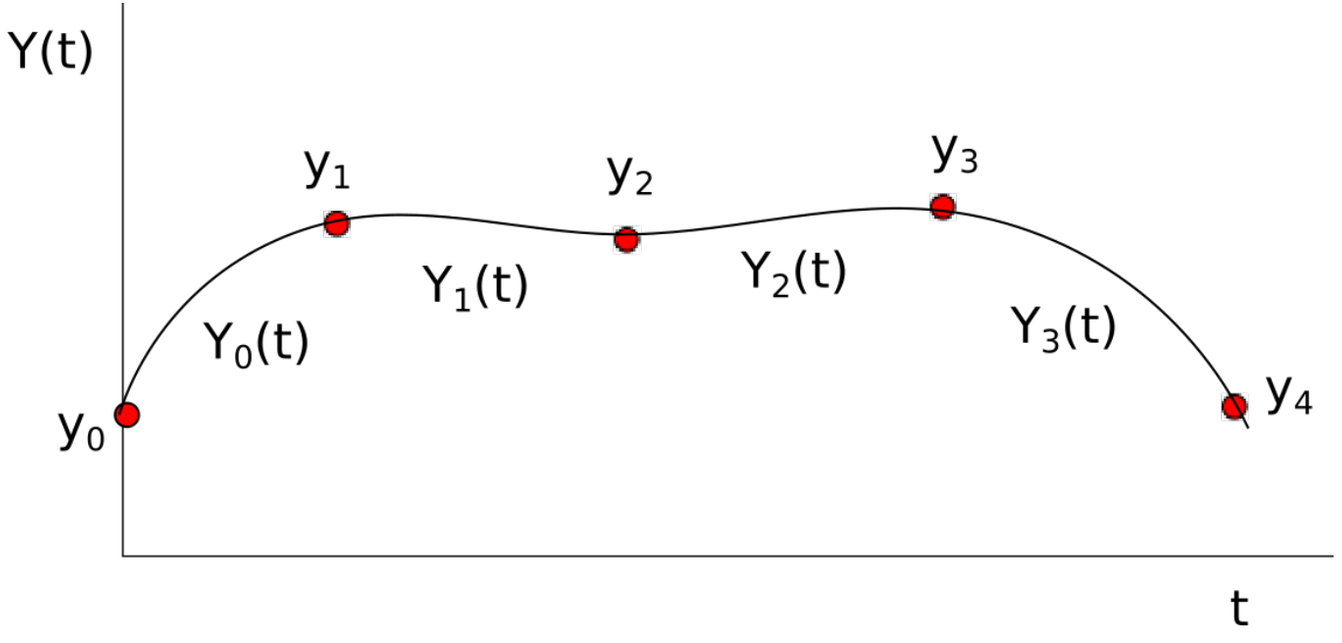

- Let us have a path given by N points in state-space $\mathbf{y}_0 \in X, \ldots \mathbf{y}_{N-1} \in X$.
- We define trajectory as pair-wise cubic function Y, i.e sequence of $N-1$ cubic functions $Y_i(t) : [0,1] \to X$ such that:
  - $Y_i(t)$ connects $\mathbf{y}_i$ with $\mathbf{y}_{i+1}$.
  - Y has continuous first and second derivatives.

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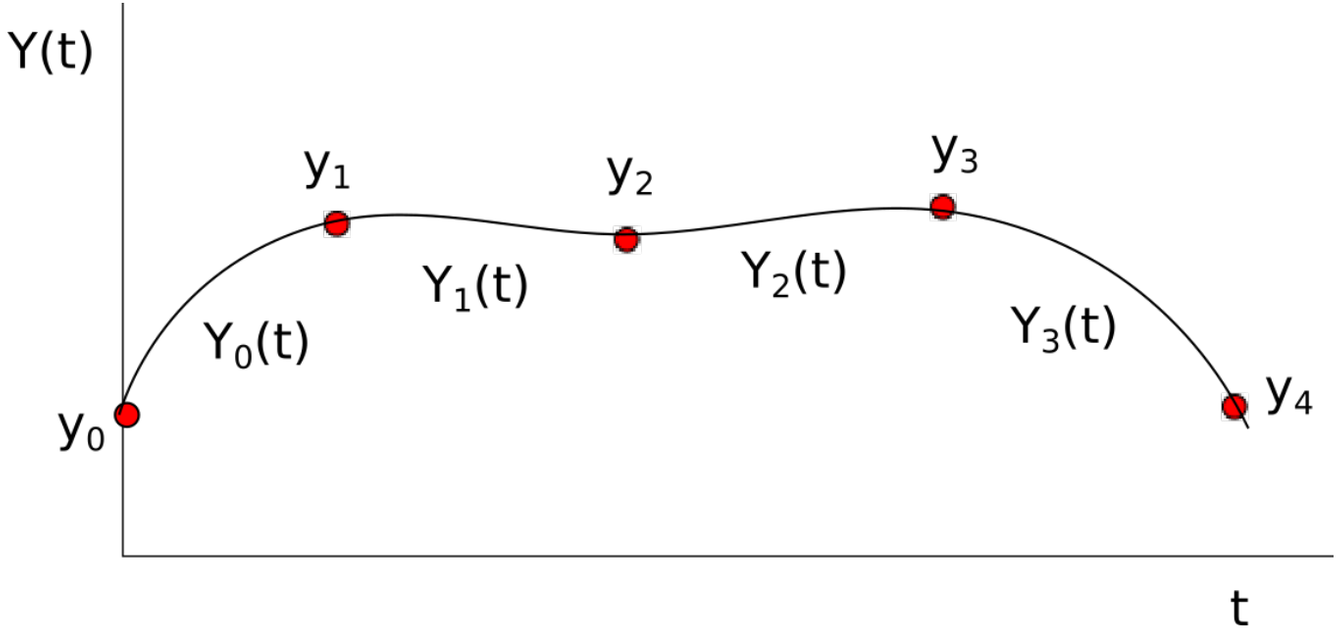

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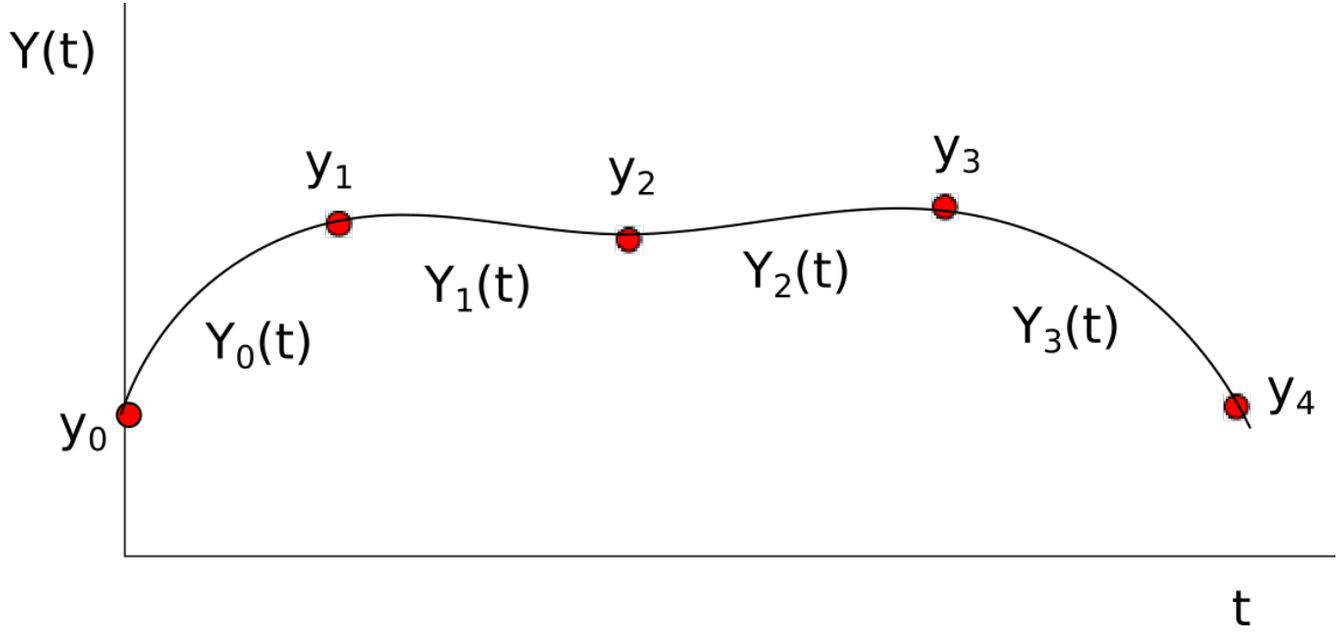

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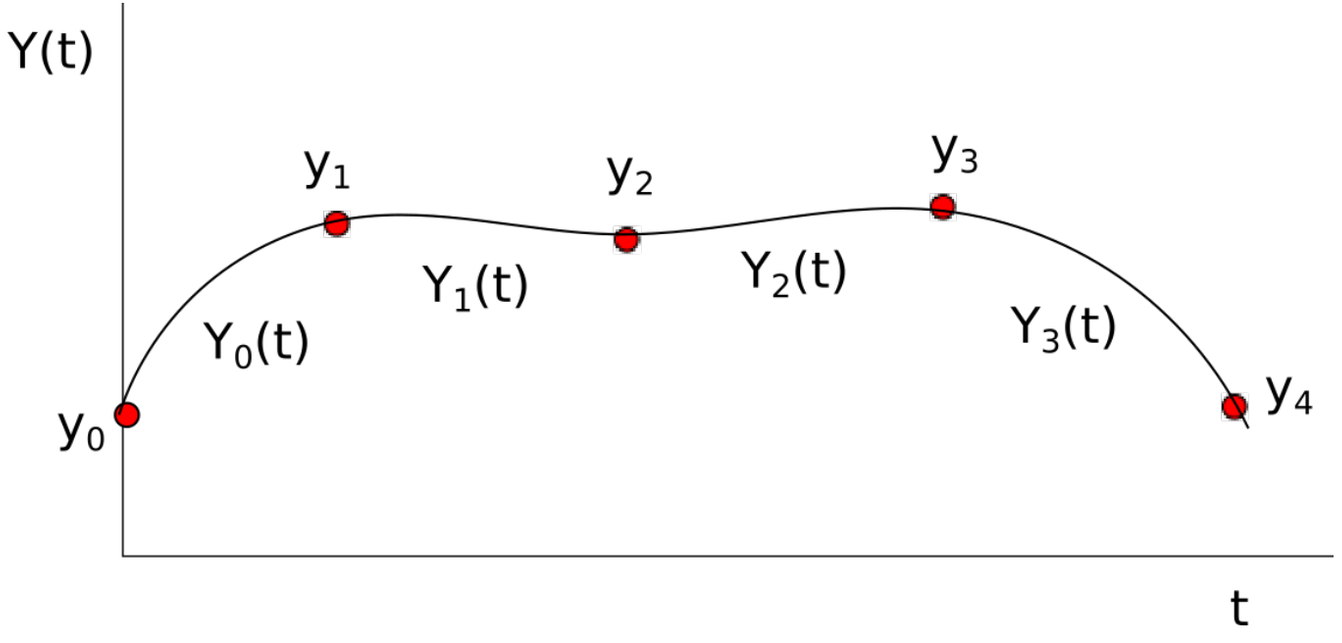

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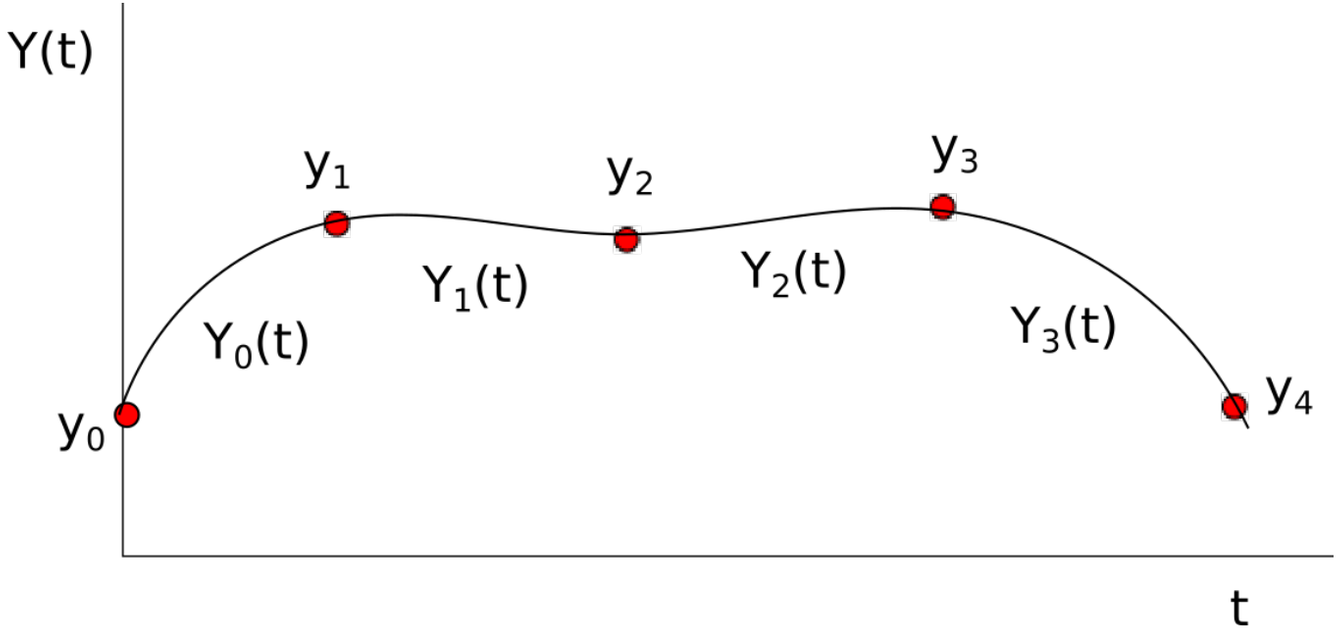

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  - $Y_i(0) = a_i = \mathbf{y}_i$
  - $Y_i(1) = b_i + c_i + d_i = \mathbf{y}_{i+1}$
  - $2 \times (N-1)$ linear equations.

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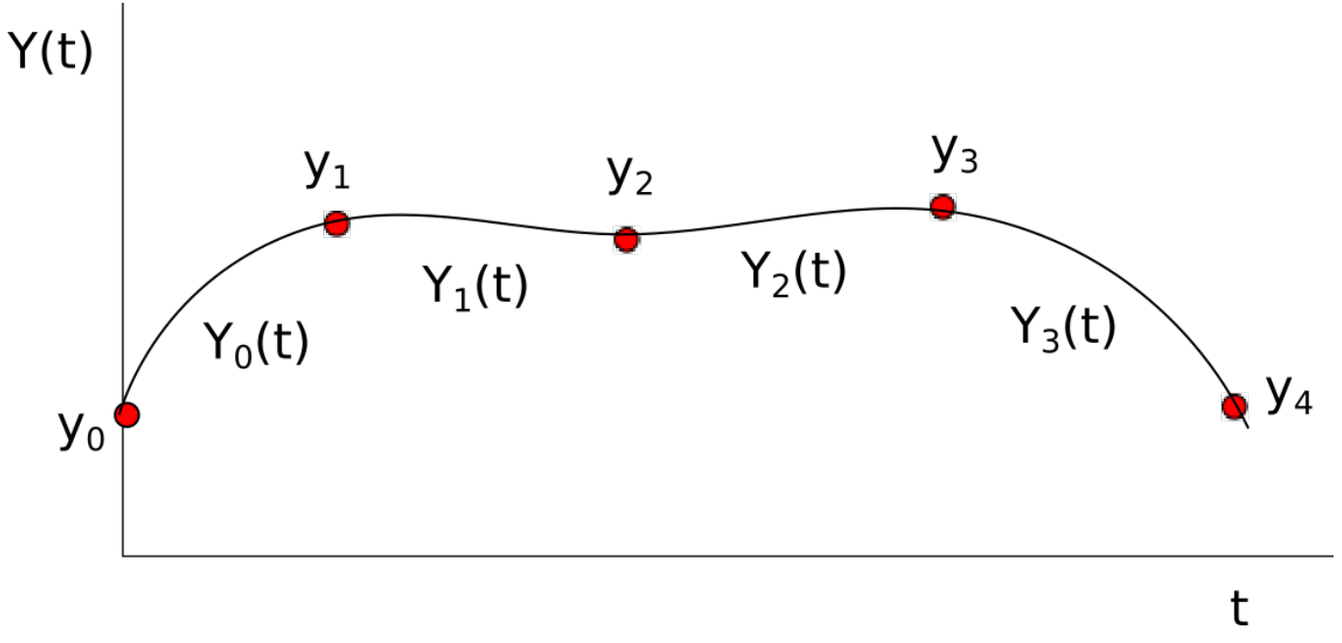

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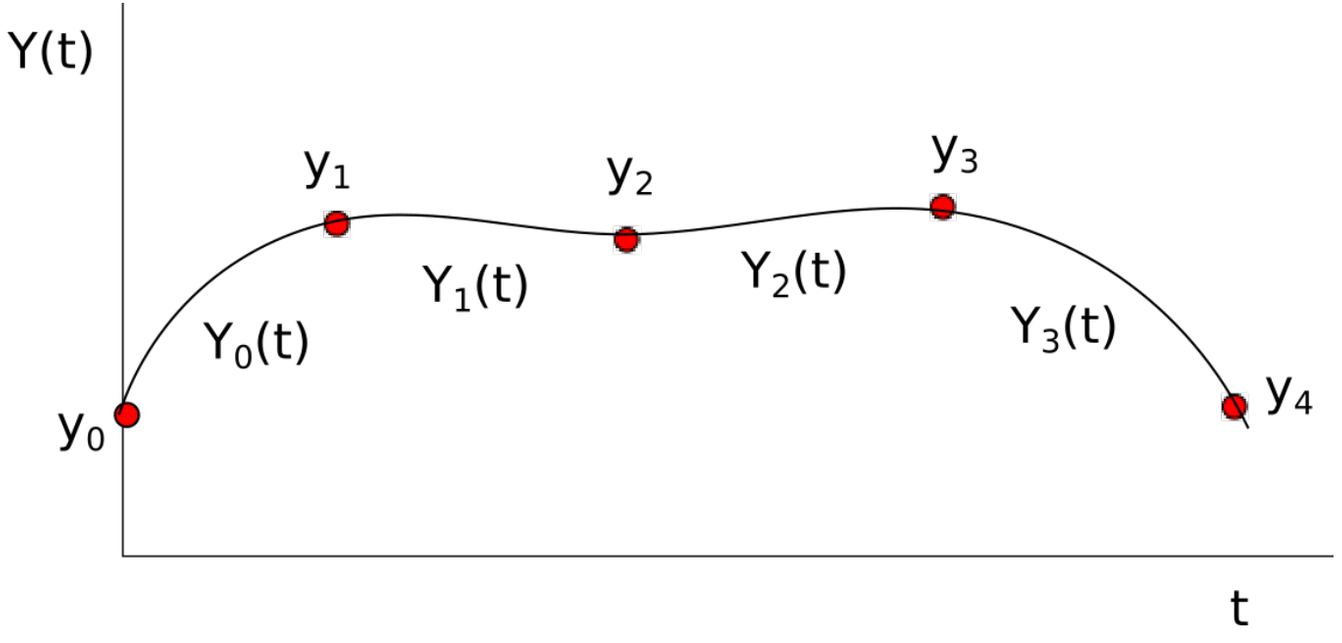

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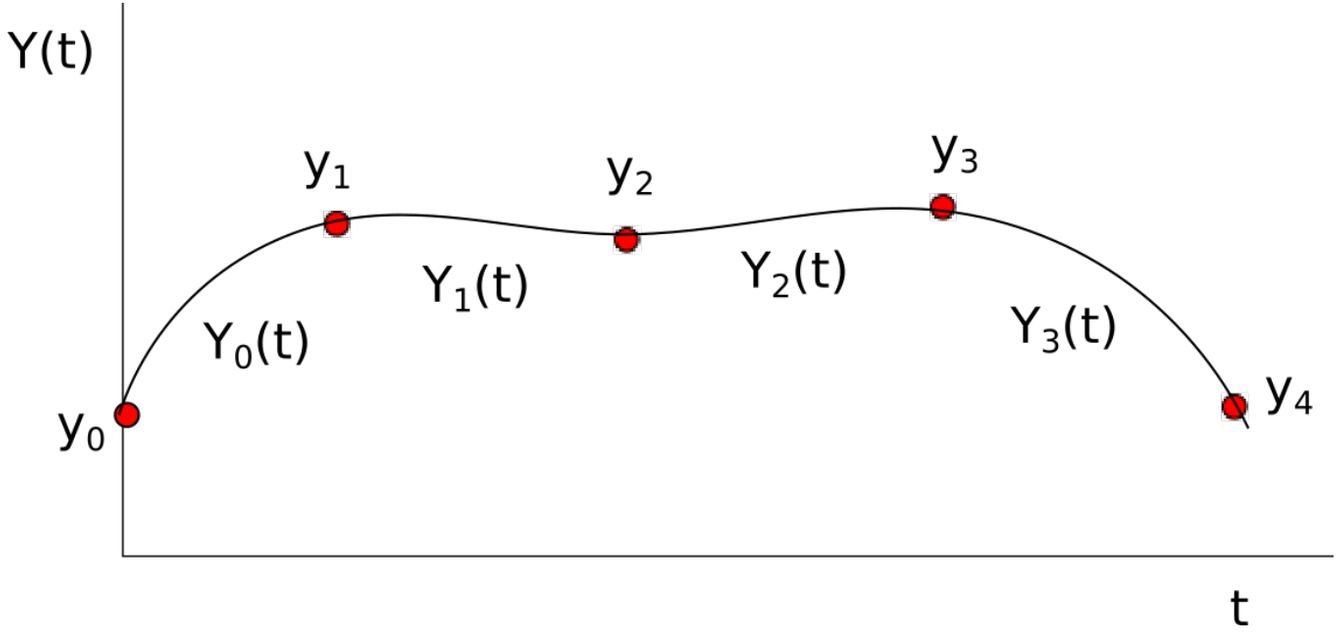

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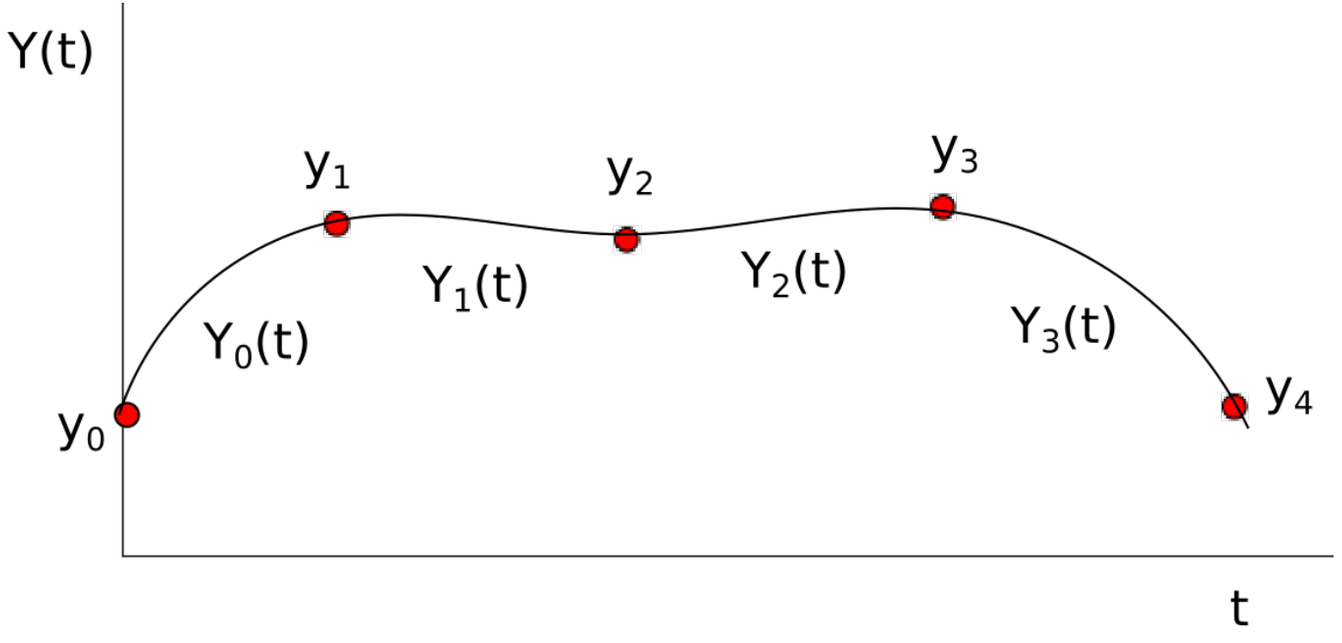

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- last two equation comes from requirements on initial and final speed, e.g.
  - $\frac{\partial Y_0(0)}{\partial t} = \frac{\partial Y_{N-1}(1)}{\partial t} = 0$

What if motion model is unknown

- Linear equations are re-arranged into $4(N-1) \times 4(N-1)$ matrix $\mathbf{A}$ and $4(N-1)$ dimensional vector right-hand side vector $\mathbf{b}$.
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- Can you design a motion control algorithm without motion model?

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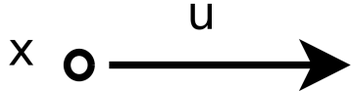

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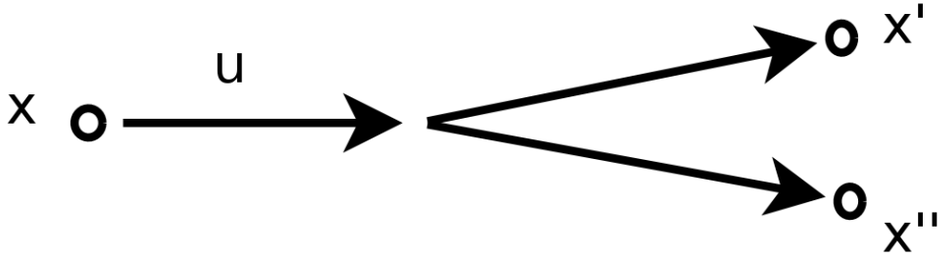

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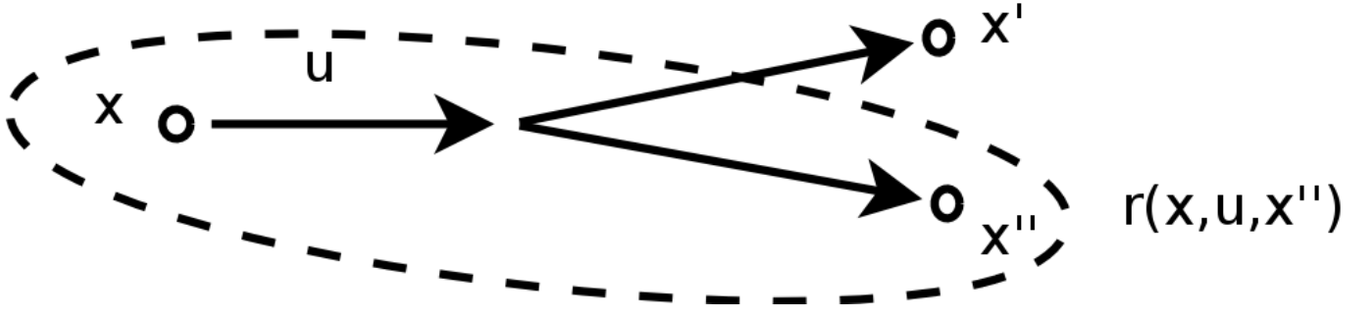

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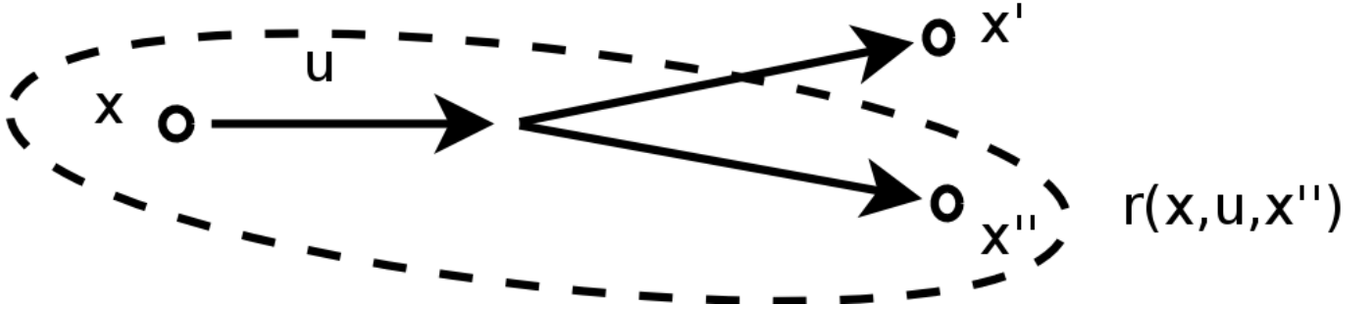
- Policy: $\pi_\theta(\mathbf{x}) : X \to U$



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- Given these trajectories, change the policy to increase expected sum of rewards

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- We solve the following optimization problem

$$\theta^* = \arg\max_\theta J(\theta)$$

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  - either solve **primal task** e.g. by following gradient ∇J to maximize $J(\theta)$ directly.
  - primal is often solved in the optimal control community (e.g. LQR),
  - or solve **dual task** to find state-action function $Q(\mathbf{x}, \mathbf{u}) : X \times U \rightarrow \mathbb{R}$ which tells expected sum rewards when choosing action $\mathbf{u}$ from state $\mathbf{x}$.
  - optimal policy $\pi^* = \arg\max_{\mathbf{u}} Q(\mathbf{x}, \mathbf{u})$
  - dual is often employed by AI community as heuristics for state-space search
  - Q-values could provide metrics for RRT [Tedrake-LQR-trees-2015]

Dual task provides alternative point-of-view (e.g. shadow prices in LP or sparse feature selection for SVM)

Primal task - approximating criterion.

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- We can approximate criterion value, what about gradient?

Primal task - approximating gradient

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- Perform several small random perturbations $\Delta\theta_i$ and compute $J(\theta + \Delta\theta_i)$.
- Relation to gradient $\nabla J(\theta)$ is given by the first order Taylor polynom

$$
\begin{aligned}
J(\theta + \Delta\theta_i) &= J(\theta) + \nabla J(\theta)^\top \Delta\theta_i \\
\Delta\theta_i^\top \nabla J(\theta) &= J(\theta) - J(\theta + \Delta\theta_i) \\
\underbrace{\begin{bmatrix} \Delta\theta_1^\top \\ \vdots \\ \Delta\theta_n^\top \end{bmatrix}}_{\text{matrix } \mathbf{A}} \nabla J(\theta) &= \underbrace{\begin{bmatrix} J(\theta) - J(\theta + \Delta\theta_1)) \\ \vdots \\ J(\theta) - J(\theta + \Delta\theta_n)) \end{bmatrix}}_{\text{vector } \mathbf{b}}
\end{aligned}
$$

Primal task - solution

- Gradient is solution of overdetermined set of linear equations:

$$\nabla J(\theta) = \begin{bmatrix} \Delta\theta_1^\top \\ \vdots \\ \Delta\theta_n^\top \end{bmatrix}^+ \cdot \begin{bmatrix} J(\theta) - J(\theta + \Delta\theta_1)) \\ \vdots \\ J(\theta) - J(\theta + \Delta\theta_n)) \end{bmatrix}$$

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- Algorithm is simple:
  - Randomly initialize θ
  - Use $\pi_\theta(\mathbf{x})$ to get trajectories.
  - Compute $\nabla J(\theta)$ using pseudo-inverse.
  - Update $\theta \leftarrow \theta + \alpha \frac{\nabla J(\theta)}{\|\nabla J(\theta)\|}$

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- Converges to local minima - good initialization needed (e.g. imitation learning).
- High-dimensional θ requires many real-world samples.
- There are better gradient approximations - see survey [2]

[2] Marc Peter Deisenroth, Gerhard Neumann and Jan Peters, A Survey on Policy Search for Robotics, NOW, 2013

Primal task - likelihood ratio trick

- Let us introduce stochastic policy:

$$\pi_\theta(\mathbf{u}|\mathbf{x}) : X \times U \to [0; 1]$$

- For example: $\pi_\theta(\mathbf{u}|\mathbf{x}) \sim \mathcal{N}(\theta^\top \mathbf{x}, \Sigma)$

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- Using likelihood ratio trick we rewrite it as follows:

$$\nabla_\theta p(\tau|\theta) = p(\tau|\theta) \nabla_\theta \log p(\tau|\theta)$$

Primal task - likelihood ratio trick

- and substitute it back to $\nabla_\theta J(\theta)$:

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- Why?

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- Why? Because $\nabla_\theta \log p(\tau|\theta)$ can be expressed using known $\pi_\theta(\mathbf{u}_k|\mathbf{x}_k)$:

$$p(\tau|\theta) = p(\mathbf{x_0}) \prod_k p(\mathbf{x}_{k+1}|\mathbf{x}_k, \mathbf{u}_k) \pi_\theta(\mathbf{u}_k|\mathbf{x}_k)$$

$$\nabla_\theta \log p(\tau|\theta) = \nabla_\theta [\, \log p(\mathbf{x_0}) + \sum_k \log p(\mathbf{x}_{k+1}|\mathbf{x}_k, \mathbf{u}_k) + \sum_k \log \pi_\theta(\mathbf{u}_k|\mathbf{x}_k) \,]$$

$$= \sum_k \nabla_\theta \log \pi_\theta(\mathbf{u}_k|\mathbf{x}_k)$$

Primal task - likelihood ratio trick

- Collect N trajectories of length M:

$$\tau_1 = [(\mathbf{u}_{1,1}, \mathbf{x}_{1,1}) \dots \mathbf{u}_{M,1}, \mathbf{x}_{M,1})]$$

$$\vdots$$

$$\tau_N = [(\mathbf{u}_{1,N}, \mathbf{x}_{1,N}) \dots \mathbf{u}_{M,N}, \mathbf{x}_{M,N})]$$

- Compute gradient (main result):

$$\nabla_\theta J(\theta) = \frac{1}{N} \sum_{i=1}^{N} \sum_{k=1}^{M} \nabla_\theta \log \pi_\theta(\mathbf{u}_{k,i} | \mathbf{x}_{k,i})$$

- update θ according to the gradient and repeat until convergence.

What you can do?

- Work with us on:
  - real Search&Rescue platform
  - better IRO tasks
- TORCS - Racing **and demolishon derby** simulator competition.
  http://en.wikipedia.org/wiki/TORCS
- Starcraft competition
  http://webdocs.cs.ualberta.ca/~cdavid/starcraftaicomp/