Bayes Decision Theory Cookbook

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- ω discrete states of the nature, categories, classes
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- $\alpha^*(\mathbf{x}) = \arg\min_\alpha R(\alpha|\mathbf{x})$ optimal strategy.

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- Optimal strategy:

$$\alpha^*(\mathbf{x}) = \arg\min_{\alpha \in \{\alpha_1, \alpha_2\}} R(\alpha|\mathbf{x}) = \begin{cases} \alpha_1 & \text{if } R(\alpha_1|\mathbf{x}) < R(\alpha_2|\mathbf{x}) \\ \alpha_2 & \text{otherwise} \end{cases}$$

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- if $\lambda_{11} < \lambda_{21}$ and $\lambda_{22} < \lambda_{12}$ then the condition is rewritten as follows:

$$\underbrace{\frac{P(\mathbf{x}|\omega_1)}{P(\mathbf{x}|\omega_2)}}_{\text{likelihood ratio}} > \underbrace{\frac{\lambda_{12} - \lambda_{22}}{\lambda_{21} - \lambda_{11}} \frac{P(\omega_2)}{P(\omega_1)}}_{\text{constant } \theta}$$

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- Then the threshold θ parametrize the decision function (classifier)

$$\alpha^*(\mathbf{x};\theta) = \begin{cases} \alpha_1 & \text{if } f(\mathbf{x}) > \theta \\ \alpha_2 & \text{otherwise} \end{cases}$$

Measuring classifier quality

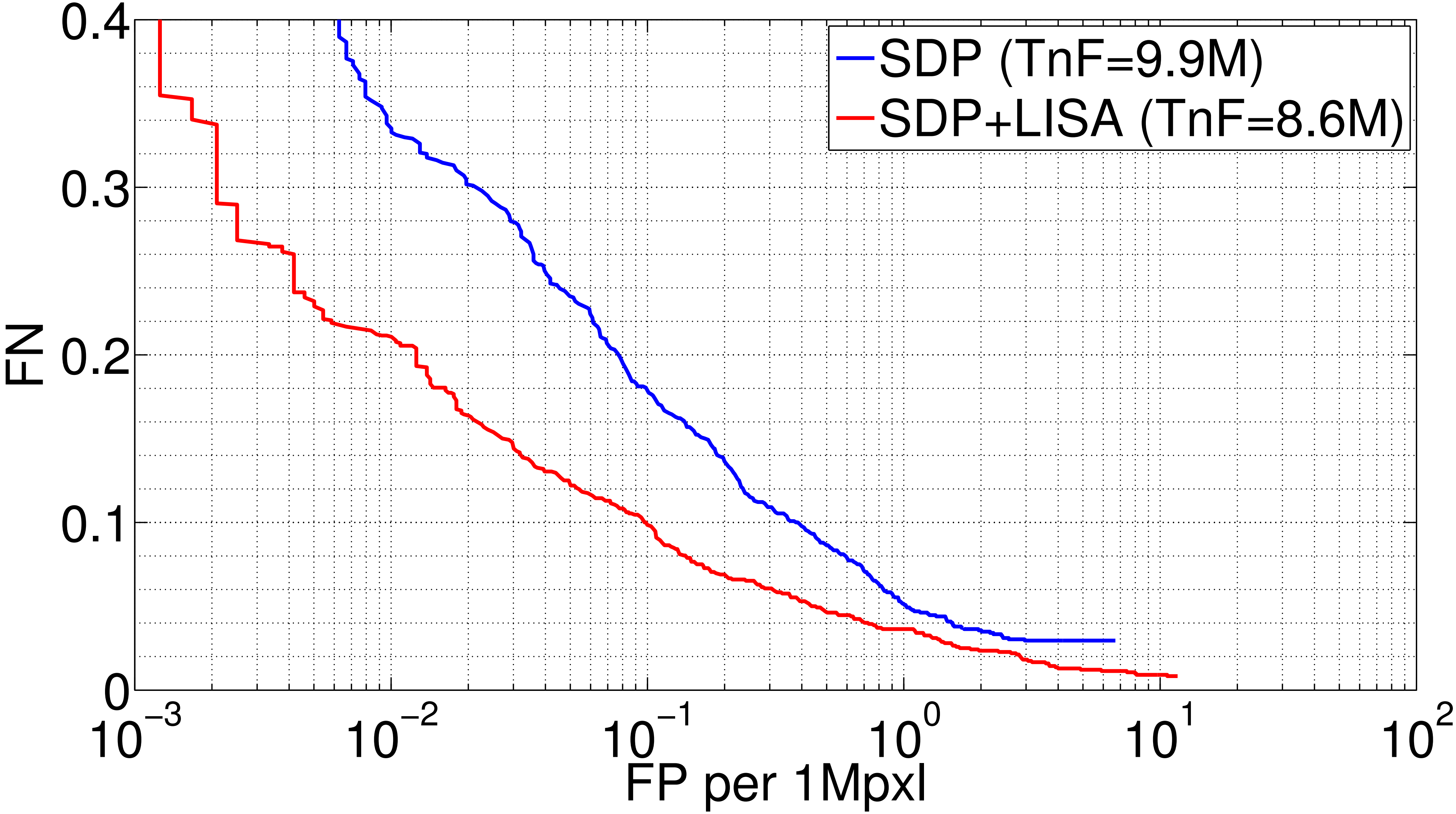
- $\omega_1/\alpha_1 \ldots$ unsafe class
  (e.g. ill patient, power-plant explosion, human detected in security camera).
- $\omega_2/\alpha_2 \ldots$ safe class
  (e.g. healthy patient, power-plant safe state, no human in security camera)
- $X_i = \{\mathbf{x} \mid \alpha^*(\mathbf{x};\theta) = \alpha_i\}$ set of all features which we classify to ω_i.
- **False negative ratio:** Probability of missing a dangerous situation (i.e the case where object is in unsafe class ω_1 and we report the safe class ω_2).

$$FN(\theta) = \sum_{\mathbf{x}\in X_2} p(\mathbf{x}|\omega_1) \approx \frac{\#\text{ of }\omega_1\text{-objects classified to }\omega_2}{\#\text{ of }\omega_1\text{-objects}}$$

- **False positive ratio:** Probability of false alarm (i.e the object is in safe class ω_2 ans we report for unsafe class α_1.

$$FP(\theta) = \sum_{\mathbf{x}\in X_1} p(\mathbf{x}|\omega_2) \approx \frac{\#\text{ of }\omega_2\text{-objects classified to }\omega_1}{\#\text{ of }\omega_2\text{-objects}}$$

Measuring classifier quality - ROC



- Which one is better?