Příklad 8.5: Najděte $I_n = \int \frac{1}{(1+x^2)^n}\,\mathrm{d}x$ na $\mathbb{R}$, pro $n = 1,\ 2$.

Poznámka: Pro $n > 1$ lze odvodit rekurentní vzorec

$$I_n = \frac{1}{2(n-1)}\left(\big(2(n-1)-1\big)\,I_{n-1} - \frac{x}{(1+x^2)^{n-1}}\right).$$

Řešení: **a)** pro $\underline{n = 1}$

$$I_1 = \int \frac{1}{1+x^2}\,\mathrm{d}x = \operatorname{arctg} x + c \qquad \text{na } \mathbb{R}.$$

b) pro $\underline{n = 2}$

$$I_2 = \int \frac{1}{(1+x^2)^2}\,\mathrm{d}x = \int \frac{(1+x^2)-x^2}{(1+x^2)^2}\,\mathrm{d}x =$$

$$= \underbrace{\int \frac{1}{1+x^2}\,\mathrm{d}x}_{\operatorname{arctg} x + c} - \int x \cdot \frac{x}{(1+x^2)^2}\,\mathrm{d}x =$$

$$= \left|\begin{array}{llll} u = x & v' = \dfrac{x}{(1+x^2)^2} & = \dfrac{1}{2} \cdot 2x(1+x^2)^{-2} \\ u' = 1 & v = \dfrac{1}{2}\,\dfrac{(1+x^2)^{-1}}{-1} & = -\dfrac{1}{2}\,\dfrac{1}{(1+x^2)} \end{array}\right| =$$

$$= \operatorname{arctg} x - \left(-\frac{1}{2}\,\frac{x}{(1+x^2)} - \int -\frac{1}{2}\,\frac{1}{1+x^2}\right) =$$

$$= \frac{1}{2} \operatorname{arctg} x + \frac{x}{2(1+x^2)} + c,$$

tj.

$$I_2 = \frac{1}{2}\left(I_1 + \frac{x}{1+x^2}\right) \qquad \text{na} \quad \mathbb{R}.$$