Digital Image

(B4M33DZO)

Lecture 2:

Fourier Transform

https://cw.fel.cvut.cz/wiki/courses/dzo/start

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PEOPLE

MARIE CURIE ACTIONS

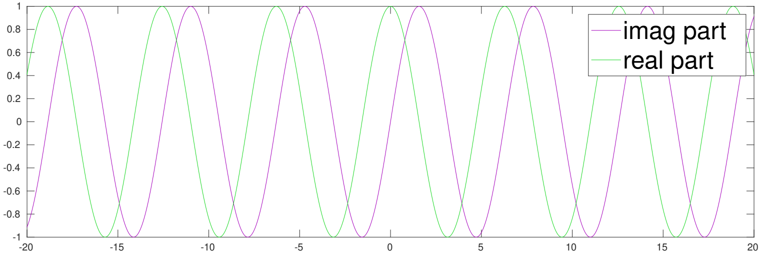
Fourier Transform

$$time\{t\} \quad \Longleftrightarrow \quad frequency\{u\}$$

$$\text{forward:} \quad F(u) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i u t}\, \mathrm{d}t$$

$$\text{inverse:} \quad f(t) = \int_{-\infty}^{\infty} F(u)\, e^{+2\pi i u t}\, \mathrm{d}u$$

$$\text{basis functions:} \quad e^{2\pi i u t} = \cos(2\pi u t) + i \sin(2\pi u t)$$



Fourier Transform

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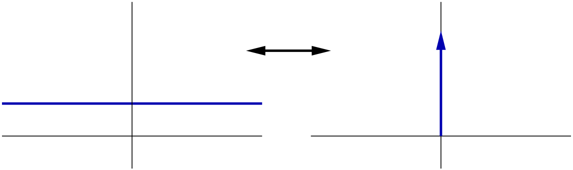
$$\text{amplitude:} \quad |F(u)| = \sqrt{\mathrm{Re}(F(u))^2 + \mathrm{Im}(F(u))^2}$$

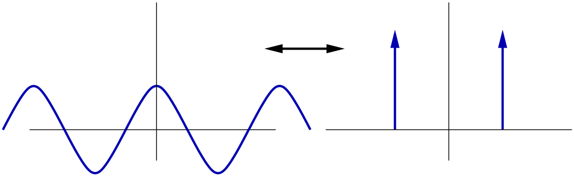
$$\text{phase:} \quad \Phi(F(u)) = \tan^{-1}\left(\frac{\mathrm{Im}(F(u))}{\mathrm{Re}(F(u))}\right)$$

Fourier Transform

$$1 \quad \Longleftrightarrow \quad \delta(u)$$

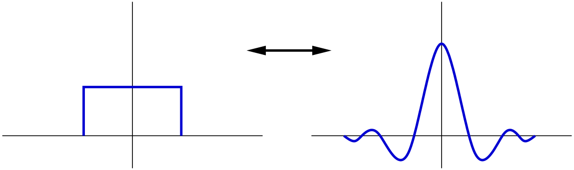


$$\cos(2\pi kx) \quad \Longleftrightarrow \quad \frac{1}{2}\Big(\delta(u+k)+\delta(u-k)\Big)$$

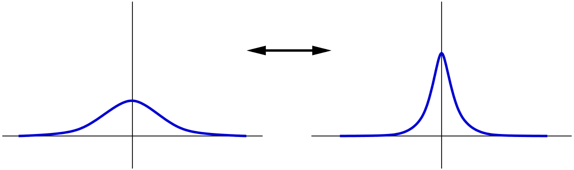


Fourier Transform

$$\mathbf{1}(x+k) - \mathbf{1}(x-k) \quad \Longleftrightarrow \quad \frac{1}{\pi u} \sin(2\pi k u)$$



$$\exp(-kx^2) \quad \Longleftrightarrow \quad \sqrt{\frac{\pi}{k}} \exp\left(-\frac{\pi^2}{k} u^2\right)$$



Fourier Transform — Properties

- **Linearity:**

$$a \cdot f(x) + b \cdot f(x) \quad \Longleftrightarrow \quad a \cdot F(u) + b \cdot F(u)$$

- **Parseval's theorem:**

$$\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x = \int_{-\infty}^{\infty} |F(u)|^2 \, \mathrm{d}u$$

- **Convolution theorem:**

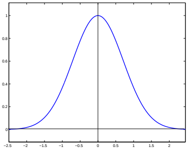
$$\int_{-\infty}^{\infty} f(x) \cdot g(t-x) \, \mathrm{d}x \quad \Longleftrightarrow \quad F(u) \cdot G(u)$$

- **Shift theorem:**

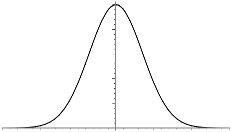
$$\int_{-\infty}^{\infty} f(x-a) \, e^{-2\pi iux} \, \mathrm{d}x = F(u) \, e^{-2\pi iua}$$

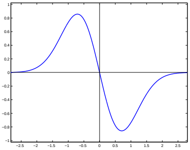
Fourier Transform — Derivatives

time | **derivatives** | **frequency**

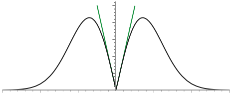


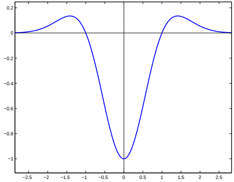
0th: $f(x) \Longleftrightarrow F(u)$



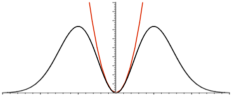


1st: $f'(x) \Longleftrightarrow 2\pi i u \cdot F(u)$





2nd: $f''(x) \Leftrightarrow (2\pi i u)^2 \cdot F(u)$



Fourier Transform (in 2D)

2D position$\{x, y\}$ $\iff$ *2D frequency coords*$\{u, v\}$
(or in polar coordinates: *frequency* (scale of waves) and *orientation* (direction of waves))

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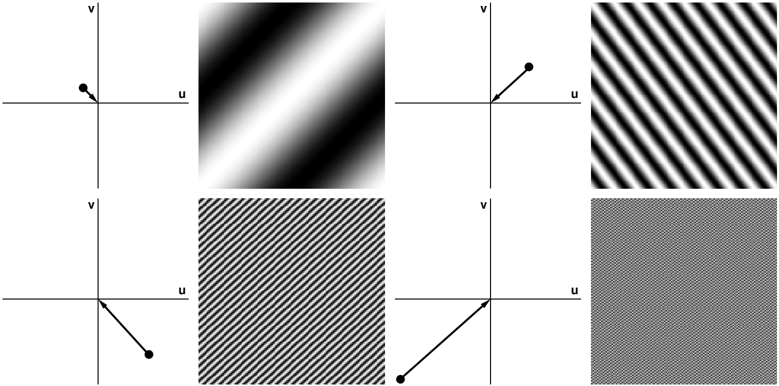
forward: $$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, e^{-2\pi i(ux+vy)}\, \mathrm{d}x\, \mathrm{d}y$$

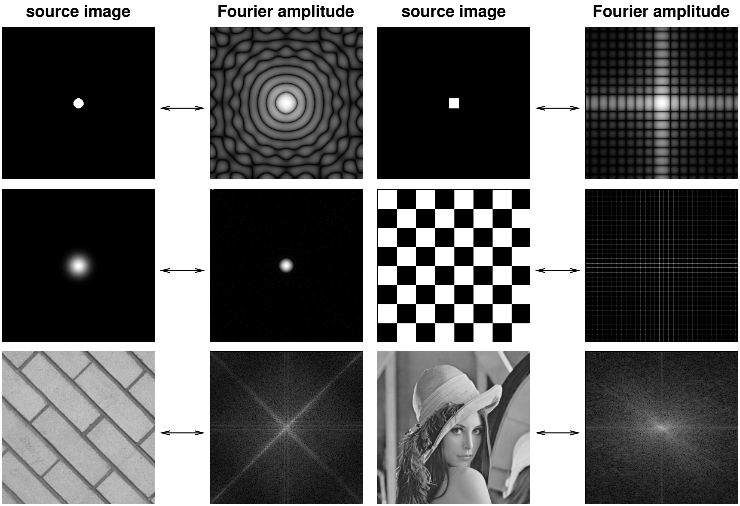
inverse: $$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v)\, e^{+2\pi i(ux+vy)}\, \mathrm{d}u\, \mathrm{d}v$$

Fourier Transform (in 2D) — Basis Functions

Basis functions (only real part shown):



Fourier Transform (in 2D)



Discrete Fourier Transform (DFT)

Discrete Fourier Transform (DFT):

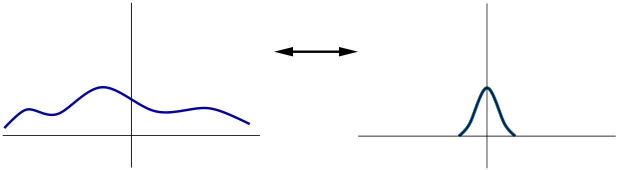
forward: $$F[u,v] = \frac{1}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f[x,y]\, e^{-2\pi i (ux+vy)/N}$$

inverse: $$f[x,y] = \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F[u,v]\, e^{+2\pi i (ux+vy)/N}$$

Sampling Theorem

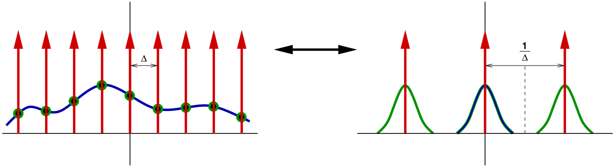
Setting:

Band-limited signal, $f_{max} \leq \frac{1}{2\Delta}$ (Nyquist frequency)



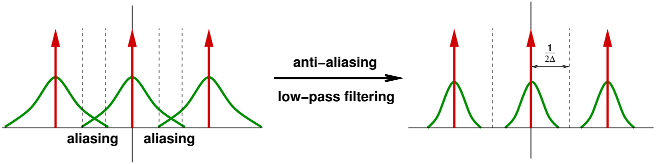
Sampling Theorem

Nyquist frequency: $f_{max} \leq \frac{1}{2\Delta}$.



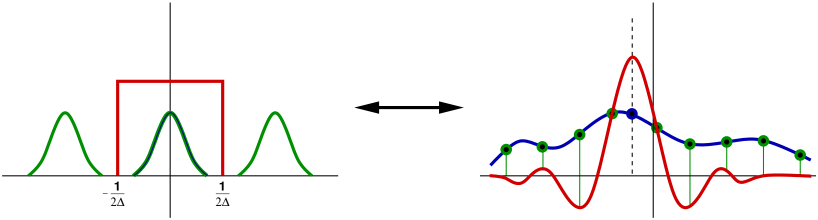
$$s(x) = \sum_k \delta(x - k\Delta) \quad \Longleftrightarrow \quad S(u) = \sum_k \delta\left(u - \frac{k}{\Delta}\right)$$

$$d(x) = f(x) \cdot s(x) \quad \Longleftrightarrow \quad D(u) = (F * S)(u)$$



Sampling Theorem — Signal Reconstruction

Signal reconstruction:



$$B(u) = \mathbf{1}\left(u + \frac{1}{2\Delta}\right) - \mathbf{1}\left(u - \frac{1}{2\Delta}\right) \quad \Longleftrightarrow \quad b(x) = \frac{1}{\pi x} \sin\left(\frac{\pi x}{\Delta}\right)$$

$$F(u) = D(u) \cdot B(u) \quad \Longleftrightarrow \quad f(x) = (d * b)(x)$$

Effect of Sampling – Example

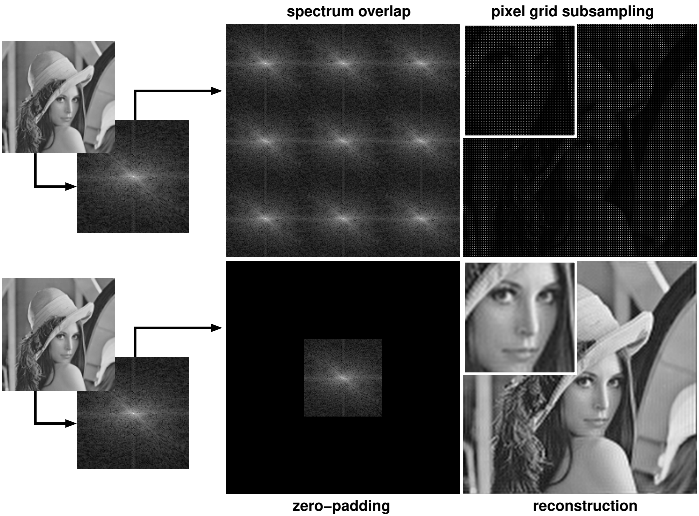
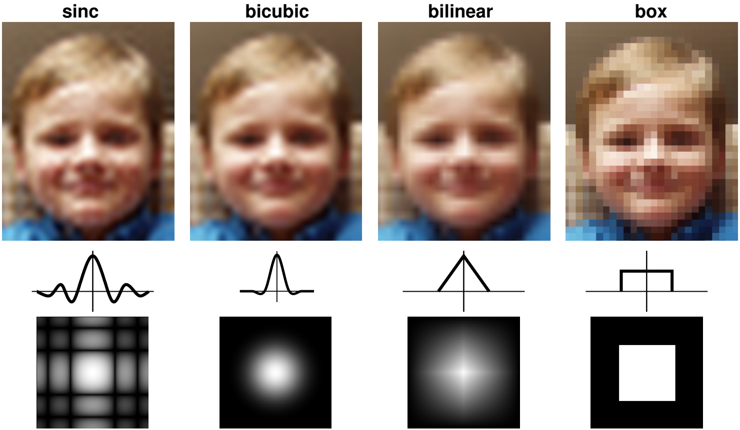


Image Resampling

Problem: convolution with sinc ⇒ time consuming, ringing artifacts.

Sinc approximations with narrow support (bicubic, bilinear, box):

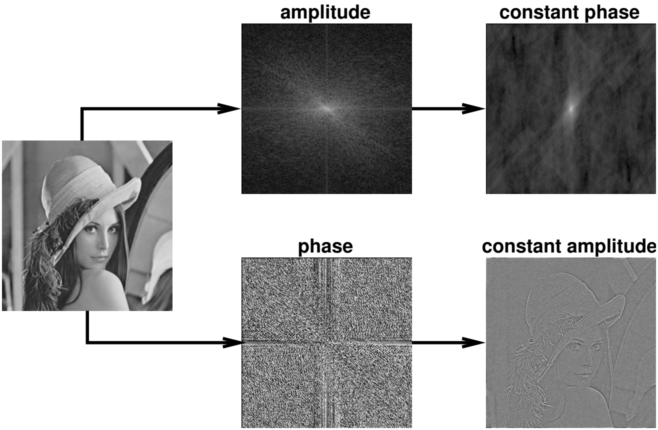


Aliasing Example

Fourier Transform — Amplitude/Phase

Edges with orientation $\arctan(v/u)$ and frequency $\sqrt{u^2 + v^2}$:

Amplitude $\Rightarrow$ intensity   Phase $\Rightarrow$ "location"



Keeping only phase (and synthesizing amplitude)

Let us keep the original phase only, and replace the amplitude by A $= f^{-\beta}$, $f = \sqrt{u^2 + v^2}$:

$\beta = 0$

$\beta = 0.4$

$\beta = 0.8$

$\beta = 1$

$\beta = 1.5$

$\beta = 2$

Fast Fourier Transform (FFT)

Computation complexity of the naive implementation is $\mathcal{O}(N^4)$.
Can we do it faster?

Computing DFT using Fast Fourier Transform (FFT):

separability: $$F[u,v] = \frac{1}{N}\sum_{x=0}^{N-1} e^{-2\pi iux/N}\left(\sum_{y=0}^{N-1} f[x,y]\, e^{-2\pi ivy/N}\right)$$

recursion:
$$F[u] = F_{even}[u] + F_{odd}[u]\, e^{-2\pi iu/N},$$
$$F[u+N/2] = F_{even}[u] - F_{odd}[u]\, e^{-2\pi iu/N}.$$

Computation complexity: $\mathcal{O}(N^2 \log N)$.