Problem solving by search
Finding the optimal sequence of states/decisions/actions

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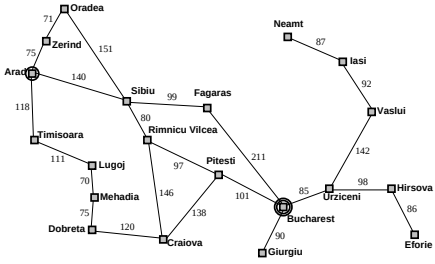
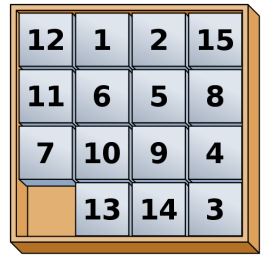
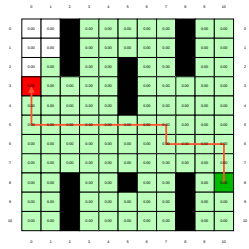
Notes

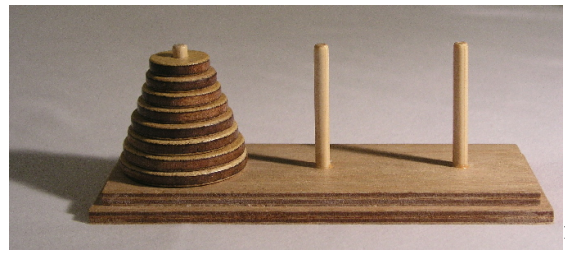
We will show that decisions/actions/control-commands are the same for deteriministic problems.

Problems to solve








[1]

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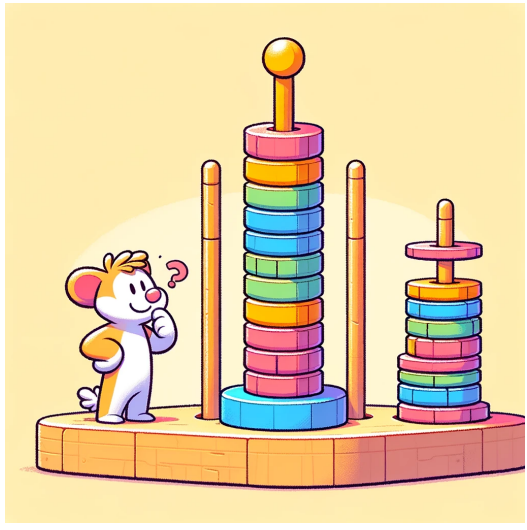
Notes

Understanding the problem is the key. DALL-E:



Notes

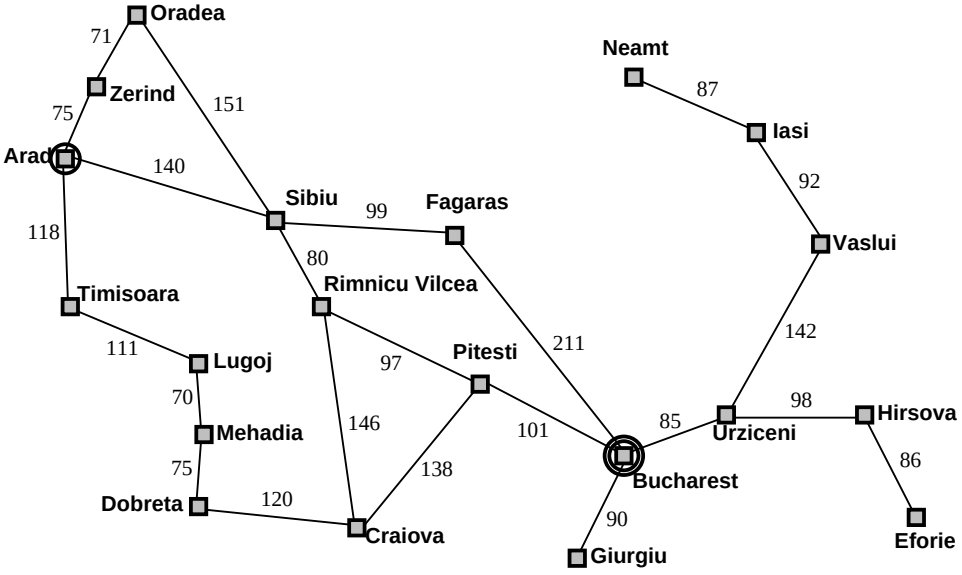
DALL-E creations after correctly explaing the problem itself.

Outline

- Search problem. *What do you want to solve?*
- State space graphs. *How do you formalize/represent the problem? Problem abstraction.*
- Search trees. *Visualization of the algorithm run.*
- Strategies: which tree branches to choose?
- Strategy/Algorithm properties. *Memory, time, ...*
- Programming infrastructure.

Notes

Example: Traveling in Romania



Notes

Ok, start with a simple one, almost everybody knows about the navigation - path planning problem. Waze, Garmin, ... Here, the problem can be transferred into a graph quite directly - a map is a kind of a graph, states are location in a city.
Can you think about more problems?
For example:

- Touring problems. Special case: Traveling salesperson problem – each city must be visited exactly once.
- Planning robot movements – mobile robot or manipulator.
- VLSI (chip) layout.
- ...

Traveling Example: States and Actions

Goal:

be in Bucharest

Problem formulation:

state: city (which city you are in)

action (decision): choice of a road

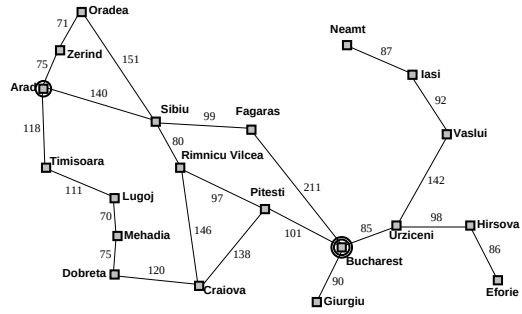
Solution:

Sequence of cities (path)

(sequence of actions/decisions [2])

Optimality – Cost, Loss, Utility, . . . :

Energy, time, tolls, . . .

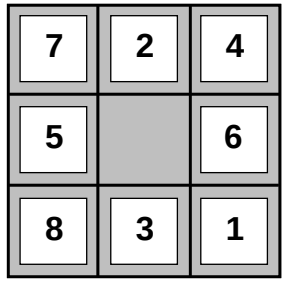


Notes

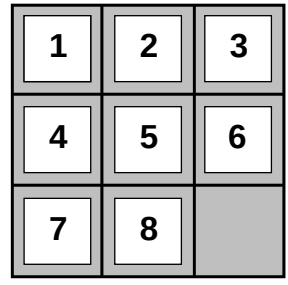
Classical problem from the Book [2], we use it, too.

states and actions will be frequently discussed in several lectures and algorithms. It is important to fully understand them. At crossings, we need to decide about the next road - this is the action. We assume that we reach the next crossing - result of the action.

Example: The 8-puzzle



Start State



Goal State

states?
actions?
solution?
cost?

Notes

- State: Location of each of the 8 tiles and the blank.
- Number of states: 9!
- Initial state: any state. (Note that any given goal state can be reached from exactly half of the initial states.)
- Actions: Movements of the blank space: Left, Right, Up, Down (or a subset of these)
- Test of solution/goal: Check whether state matches the goal configuration.
- Path cost: nr. steps in the path (each step costs 1)

Toy problem (3.2.1) from [2].

A Search Problem

- State space (including initial state): position, board configuration, ...
- Action space/set : drive to, Up, Down, Left, ...
- Transition model : Given a state and an action, return a result state (and cost or reward of the transition)
- Goal test : Are we done?

Notes

We will use the terminology throught the next 5-6 lectures; also for Markov (Sequential) Decision Processes, Reinforcement Learning.

Make a mental test: You are a robot, going from home to school. What would be states, actions, transition model, goal test?

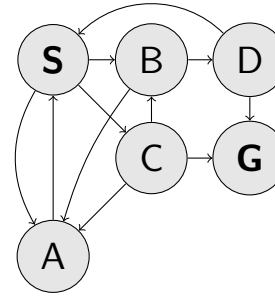
Transition model can be also understood as a mapping between actions and results/outcome.

Discrete State Space

State space graph: a representation of a search problem

- States $s \in \mathcal{S} = \{\mathbf{S}, \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathbf{G}\}$ (finite set)
- Arcs represent actions a: for each state s, $a \in \mathcal{A}(s)$ ($\mathcal{A}$ is also finite)
- State transition function $s' = \mathrm{result}(s, a)$
- Start (initial) state $s_0 \in \mathcal{S}$, $s_0 = \mathbf{S}$.
- Set of goal states $\mathcal{S}_G \subset \mathcal{S}$.

Each state occurs only *once* in a state (search) space.

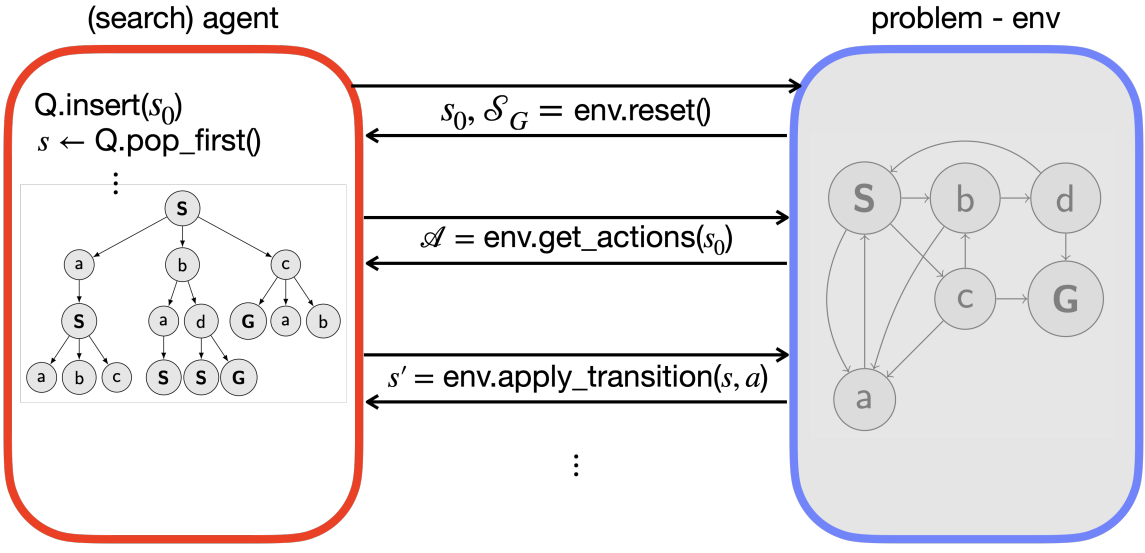


Notes

Formalizing a real world problem – (creating) a state space graph – could be a problem in itself. I put creating into brackets as it may be also infinite.
Close connection to graph algorithms like Dijkstra, Floyd-Warshall.

- Graph algorithms assume complete info about the graphs - the main input.
- For many real-world problems, the graph is not known in advance.
- The state space graph is *revealed during the search*. The graph serves as an abstraction - mental model - rather than as an actual data representation.
- Many real world problems have too many vertices. Think about $n - 1$ puzzle or chess: the number of possible configurations is enormous.
- A solution can be actually quite shallow.

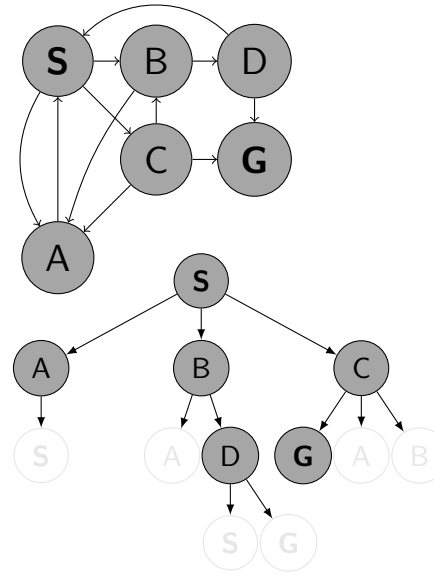
Agent–Problem Dialog (a programmer's viewpoint, uknown graph)



Notes

BFS: Q is FIFO data structure (queue)

```
1:  function FORWARD_SEARCH
2:      Q.insert(_, s_0) and mark s_0 as visited
3:      while Q not empty do
4:          p, s ← Q.pop()
5:          parent[s] ← p
6:          if s ∈ S_G then return Success
7:          for all a ∈ A(s) do
8:              s' ← result(s, a)
9:              if s' not visited then
10:                 Mark s' as visited
11:                 Q.insert(s, s')
12:             else
13:                 Resolve duplicate s' in Q
        return Failure
```



Q: (_,**S**) (S,A) (S,B) (S,C) (B,D) (C,**G**)
visited: **S** A B C D **G**

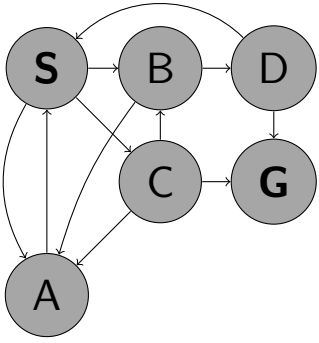
Notes

- What does the Q.pop() function/method do?
- Do we need to resolve duplicates somehow? If not, why?
- Could we stop and report success earlier?
- How to create the path?

This is the key slide to understand the difference between graph problem and tree search.
Create the search tree by pencil, think about Q and visited (whatever it may be).
How would you name the data structure Q? What kind of data structure?
When building the search tree:

- white - result of transition from a given state using a give action
- gray - visited and inside Q
- dark gray - visited and explored (outside Q)
- (made) invisible - thrown away

Search algorithm partitions state space into 3 disjoint sets



Can you find good names for them?

Notes

Search (algorithm) properties

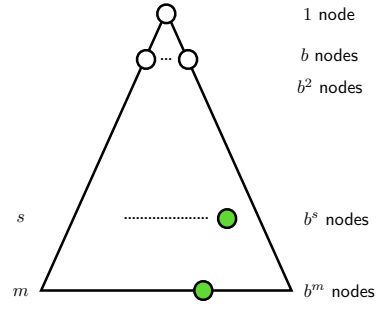
- Guaranteed to find a solution (if exists)? Complete?
- Guaranteed to find the least cost path? Optimal?
- How many steps (operations with a node) does it need? Time complexity?
- How many nodes to remember? Space/Memory complexity?

How many nodes in a (search) tree? What are tree parameters?

Notes

Sketch a (search) tree (symbolicly, as a triangle). It may grow upwards or downwards. How would you characterize/parametrize *size* of a tree.

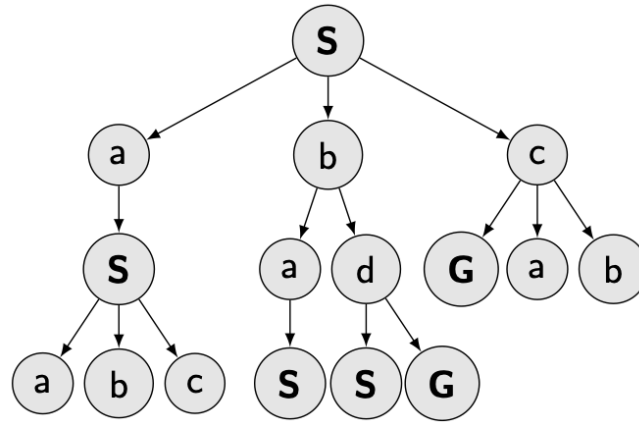
- Depth d of a node in the tree.
- Max-Depth of the tree m. Can be ∞.
- (Averege) branching factor b.
- s denotes the depth of the shallowest Goal.
- How many nodes in the whole tree?

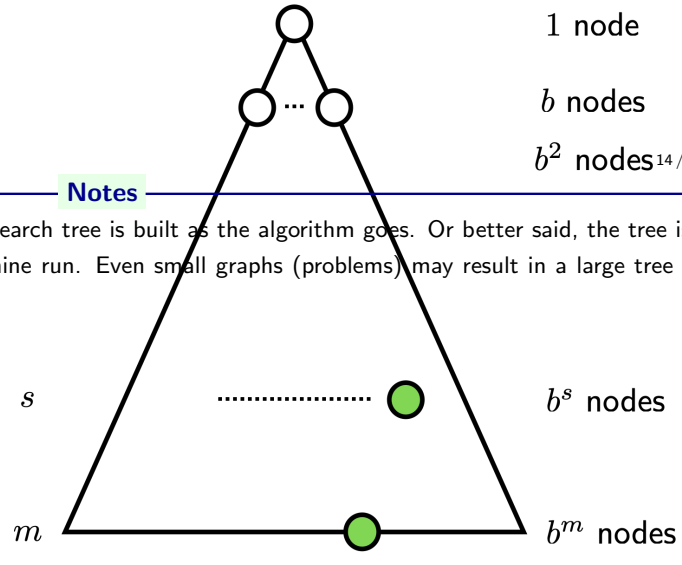


Search Strategies

How to traverse/build a search tree?

- Depth d of a node in the tree.
- Max-Depth of the tree m. Can be ∞.
- (Averege) branching factor b.
- s denotes the depth of the shallowest Goal.
- How many nodes in the whole tree?





Notes

It is perhaps worth to remember that the search tree is built as the algorithm goes. Or better said, the tree is a human friendly representation of the machine run. Even small graphs (problems) may result in a large tree - depending on the search algorithm.

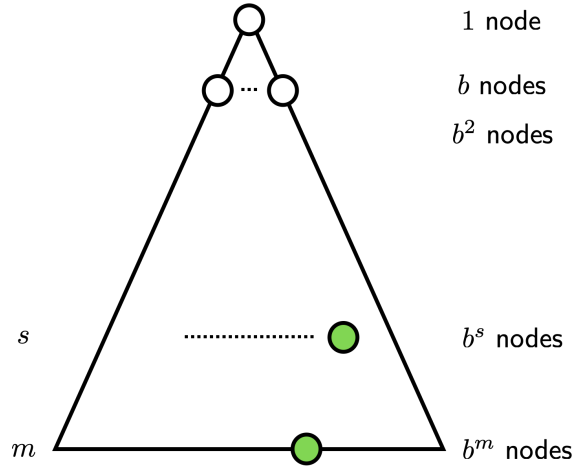
BFS properties

Complete?
Optimal?
Time complexity?

- **A** $\mathcal{O}(bm)$
- **B** $\mathcal{O}(b^m)$
- **C** $\mathcal{O}(m^b)$
- **D** $\mathcal{O}(b^s)$

Space complexity?

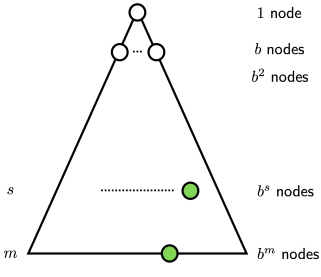
- **A** $\mathcal{O}(bm)$
- **B** $\mathcal{O}(b^m)$
- **C** $\mathcal{O}(m^b)$
- **D** $\mathcal{O}(b^s)$



Notes

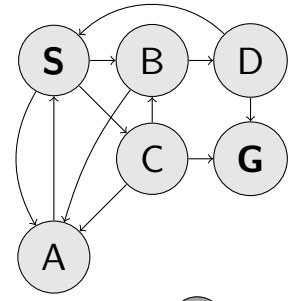
- Time, can process the whole tree until s: b^s. Well, actually $b + b^2 + b^3 + \cdots + b^s$, but the last layer vastly dominates. Try some calculations for various b.
- Space, all the frontier: b^s
- Completness: Yes!
- Optimality: it does not miss the shallowest solution, hence if all the transition costs are 1: Yes!

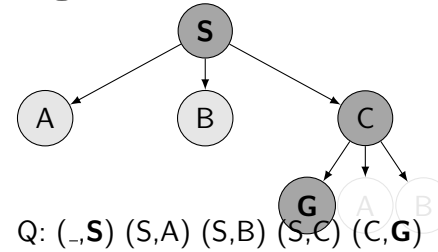
Think about the Complexities in terms of $|\mathcal{S}|$ and $|\mathcal{A}|$ (Graph theory: vertices, edges)



DFS: Q is LIFO data structure (stack)

```
1: function FORWARD_SEARCH
2:     Q.insert(_, s_0) and mark s_0 as visited
3:     while Q not empty do
4:         p, s ← Q.pop()
5:         parent[s] ← p
6:         if s ∈ S_G then return Success
7:         for all a ∈ A(s) do
8:             s' ← result(s, a)
9:             if s' not visited then
10:                Mark s' as visited
11:                Q.insert(s, s')
12:            else
13:                Resolve duplicate s' in Q
       return Failure
```





Q: (_,**S**) (S,A) (S,B) (S,C) (C,**G**)
visited: **S** A B C **G**

Notes

Do we need to resolve duplicates somehow? If not, why?

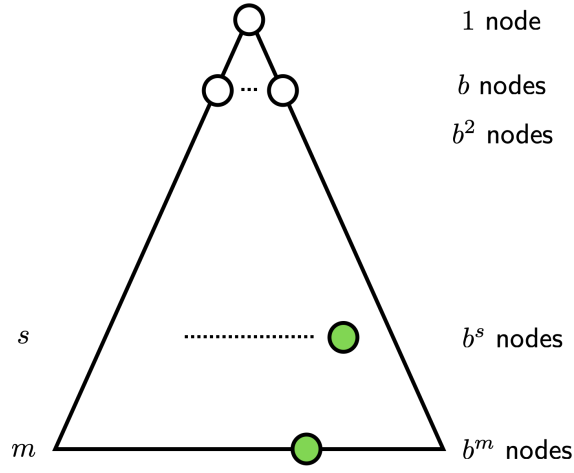
DFS properties

Complete?
Optimal?
Time complexity?

- **A** $\mathcal{O}(bm)$
- **B** $\mathcal{O}(b^m)$
- **C** $\mathcal{O}(m^b)$
- **D** $\mathcal{O}(b^s)$

Space complexity?

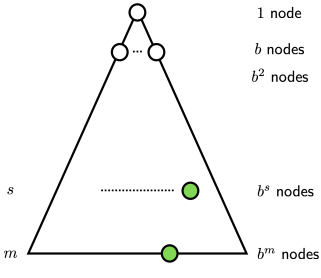
- **A** $\mathcal{O}(bm)$
- **B** $\mathcal{O}(b^m)$
- **C** $\mathcal{O}(m^b)$
- **D** $\mathcal{O}(b^s)$



Notes

- Time: can process the whole tree, b^m
- Space: only the path so far, bm (a path from root to leaf (m), plus siblings on the path are also on the frontier ($b \times m$))
- Completness: m may be ∞ hence, not in general
- Optimality: No! It just takes the first solution found.

Think about the Complexities in terms of $|\mathcal{S}|$ and $|\mathcal{A}|$ (Graph theory: vertices, edges)



Iterative deepening DFS (ID-DFS)

- Start with `maxdepth = 1`
- Perform DFS with limited depth. Solution found?
- If solution not found, forget everything, increase `maxdepth` and go to the previous step.

Isn't it terribly wasteful to forget everything between steps?

Notes

Really, how much do we repeat/waste? The "upper levels", close to the root, are repeated many times. However, in a tree, most nodes are the bottom levels and nr. nodes traversed is what counts. More specifically, for a solution at depth s, the nodes on the bottm level are generated only once, those on the next-to-bottom level 2x ... children of the root are generated $s\times$. Compare the number of nodes generated ID-DFS vs. BFS:

$$N(\text{ID-DFS}) = (s)b + (s-1)b^2 + (s-2)b^3 + \cdots + (1)b^s$$

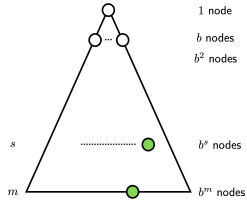
$$N(\text{BFS}) = b + b^2 + b^3 + \cdots + b^s$$

Try some calculations for various s and b. For $b = 10$ and $d = 5$:

$$N(\text{ID-DFS}) = 50 + 400 + 3000 + 20000 + 100000 = 123450$$

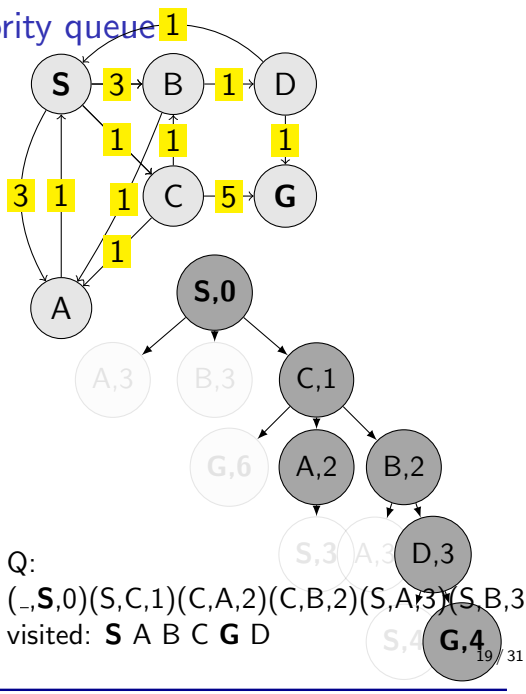
$$N(\text{BFS}) = 10 + 100 + 1000 + 10000 + 100000 = 111110$$

(Example from [2].)



Uniform Cost Search (Dijkstra): Q is priority queue

```
1: function FORWARD_SEARCH
2:     Q.insert(_, s_0, 0) and mark s_0 as visited
3:     while Q not empty do
4:         p, s, _ ← Q.pop_first()
5:         parent[s] ← p
6:         if s ∈ S_G then return Success
7:         for all a ∈ A(s) do
8:             s', c ← result(s, a)          ▷ c is cost
9:             if s' not visited then
10:                Mark s' as visited
11:                Q.insert(s, s', cost_from_start)
12:            else
13:                Resolve duplicate s' in Q
       return Failure
```



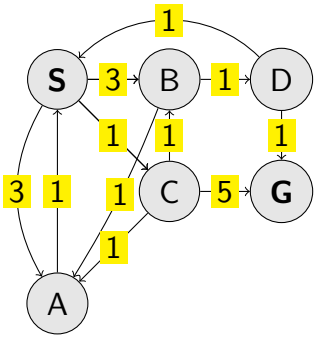
Q:
(_,**S**,0)(S,C,1)(C,A,2)(C,B,2)(S,A,3)(S,B,3)(

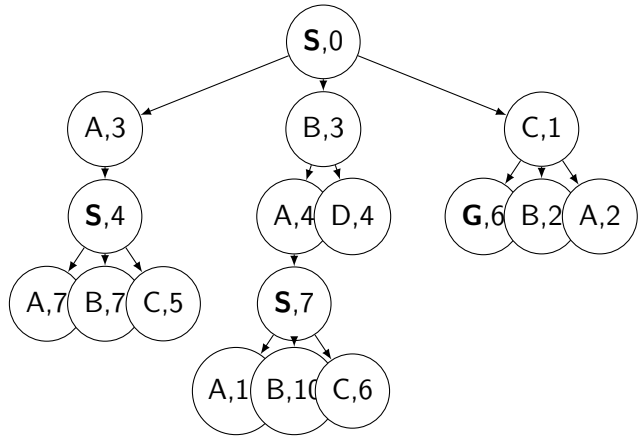
visited: **S** A B C **G** D

Notes

- Do we need to resolve duplicates somehow? If not, why?
- How is the `cost_from_start` computed?
- Why is it (sometimes) called Uniform cost search?

UCS properties





Complete?
Optimal?
Complexities?

Notes

Parts of the (complete) search tree repeat, but with different costs

Node selection, take argmin $f(n)$. Search Node: $n = (p, s, \text{cost\_value})$

Selecting next node to explore (pop operation):

$$\texttt{node} \leftarrow \underset{n \in \texttt{Q}}{\operatorname{argmin}} f(n)$$

What $f(n)$ do DFS, BFS, and UCS use? Find the right pairs!

- DFS:
- BFS:
- UCS:

- $f(n) = n.\text{cost\_from\_start}$
- $f(n) = n.\text{depth}$
- $f(n) = -n.\text{depth}$

The good: priority queue as a universal data structure for frontier
(Still, stack (LIFO) and queue (FIFO) are (conceptually) the perfect data structures for DFS and BFS, respectively.)
The bad: all the $f(n)$ correspond to the accumulated cost from start to n, cost_from_start .
What is missing?

Notes

- DFS: $f(n) = -n.depth$
- BFS: $f(n) = n.depth$
- UCS: $f(n) = n.path\_cost$

Do humans look back when planing path? Is looking back important at all? If yes, when?

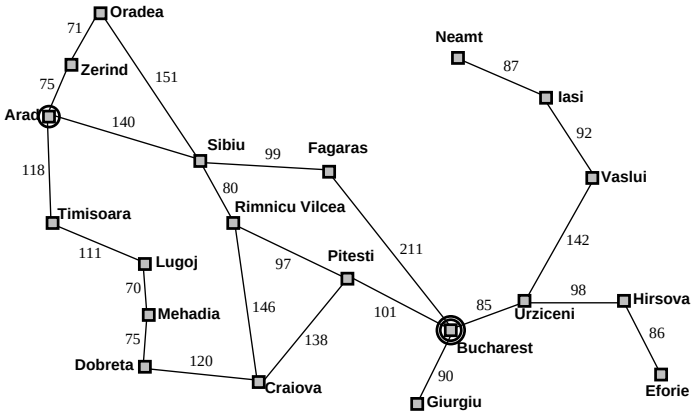
How close to a goal are we, cost-to-go ? – Heuristics

- A function that estimates how close a *state* is to the closest goal.
  ("What cost do we still have to pay to get to any of the goals?")
- Designed for a particular problem.
- $h(s)$ – it is a function of a state (its value becomes an attribute of the search node)
- Applied to a *node* n, $h(n)$ is the heuristic value of a state of node n.

Notes

What happens if $h(s) =$ true cost of the cheapest path to a goal?

Example of heuristics



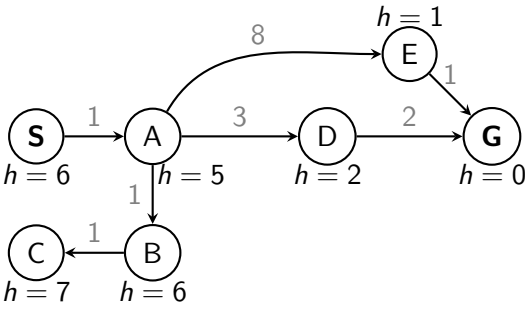
Arad	366	Mehadia	241
Bucharest	0	Neamt	234
Craiova	160	Oradea	380
Drobeta	242	Pitesti	100
Eforie	161	Rimnicu Vilcea	193
Fagaras	176	Sibiu	253
Giurgiu	77	Timisoara	329
Hirsova	151	Urziceni	80
Iasi	226	Vaslui	199
Lugoj	244	Zerind	374

Notes

Straight-line distance to Bucharest.

Illustration of *greedy* failing: Imagine going from Iasi to Fagaras. Neamt will be chosen for expansion. This will add Iasi back. Iasi is closer to Fagaras than Vaslui is and will be expanded again. Infinite loop... (3.5.1. in [2])

Greedy, take the $n^* = \text{argmin}_{n \in \mathbb{Q}} h(n)$



What is wrong (and nice) with the Greedy?

Notes

Also called "Greedy best-first search" [2].
What will happen in this example?

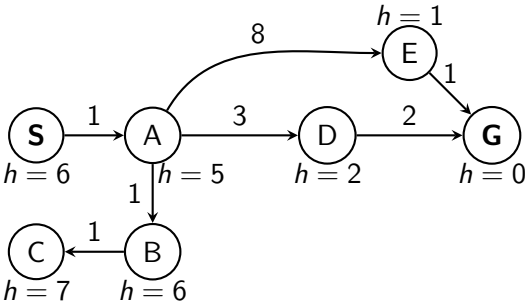
1. Expand "S". Add "A" to frontier.
2. Expand "A". Add "B", "D", "E".
3. Expand "E" ($h = 1$). Get "G".

Bad:

- not optimal
- not complete (tree search version; can be shown on the Romania example – go back)
- graph search version is complete only in finite state spaces

Nice: it is simple.

A* combines UCS and Greedy



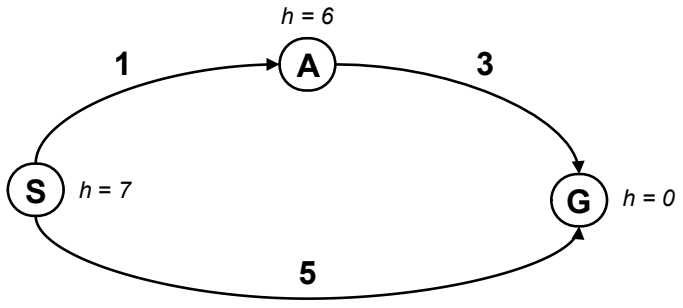
UCS orders nodes by $g(n)$ (`cost_from_start`)
Greedy orders nodes by $h(n)$ (heuristics, `cost_to_go`)

A* orders nodes by: $f(n) = g(n) + h(n)$

Notes

Trace the search algorithm on the paper. Does it find the shortest path?

Is A* optimal?



[2]

What is the problem?

[2] Graph example: Dan Klein and Pieter Abbeel

Notes

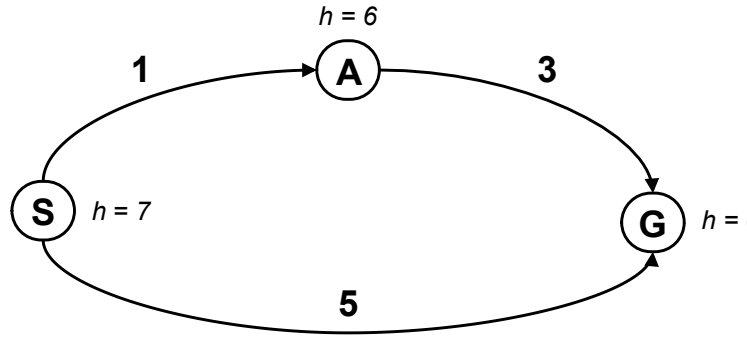
Try to answer the question before going to the next slide.

1. S
   - $f(S) = g(S) + h(S) = 0 + 7 = 7$
   - expanding/poping this one and crossing out (removing from frontier)
2. $S \to A$
   - $f(A) = g(A) + h(A) = 1 + 6 = 7$
3. $S \to G$
   - $f(G) = g(G) + h(G) = 5 + 0 = 5$
   - This is now cheapest on the frontier. I pop/expand and I'm done.

Ooops! That's not cheapest! What went wrong?
What follows – keep for next slide.
Problem with $h(A) = 6$. Overestimating the expense. (Same problem for $h(S)$.)
Estimates need to be $\leq$ actual costs.

What is the right $h(A)$?

A: $0 \leq h(A) \leq 4$
B: $h(A) \leq 3$
C: $0 \leq h(A) \leq 3$
D: $0 \leq h(A)$



Notes

$h(A) \leq 3$ it means less than the actual cost of the cheapest path from A to goal. Heuristic must not be overly pesimistic.
B is correct.
Negative $h(n)$ does not break the admissibility property but $h(\text{Goal}) = 0$ must be kept, always.
For a discussion, see, e.g.
https://stackoverflow.com/questions/30067813/are-heuristic-functions-that-produce-negative-values-inadmissible

Admissible heuristics

A heuristic function h is admissible if:

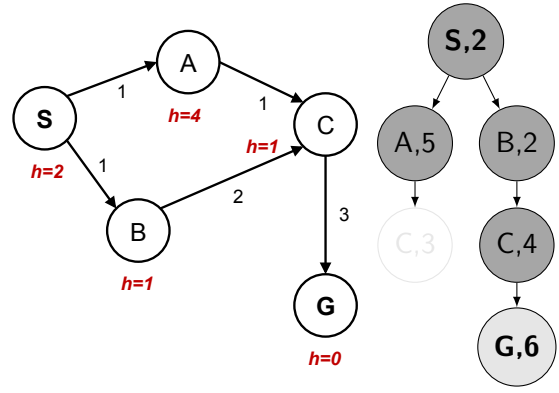
$$\begin{aligned} \forall \text{Goal} : h(n) &\leq \text{cost}(n.state, \text{Goal}) \\ h(\text{Goal}) &= 0 \end{aligned}$$

Notes

Even if negative heuristic value is allowed on the way to goal, does it make sense? How would you intepret $h(n) = 0$? Is it a meaningful minimum? Why?

Heuristics: Is admissibility sufficient?

```
1: function FORWARD_SEARCH
2:     Q.insert(_, s_0, 0, h(s_0))
3:     Mark s_0 visited
4:     while Q not empty do
5:         p, s, g, _ ← Q.pop()
6:         parent[s] ← p
7:         if s ∈ S_G then return Success
8:         for all a ∈ A(s) do
9:             s', c ← result(s, a)
10:            if s' not visited then
11:                Mark s' as visited
12:                Q.insert(s, s', g + c, g + c + h(s'))
13:            else
14:                Resolve duplicate s' in Q
    return Failure
```



Notes

What would be the proper $h(A)$?

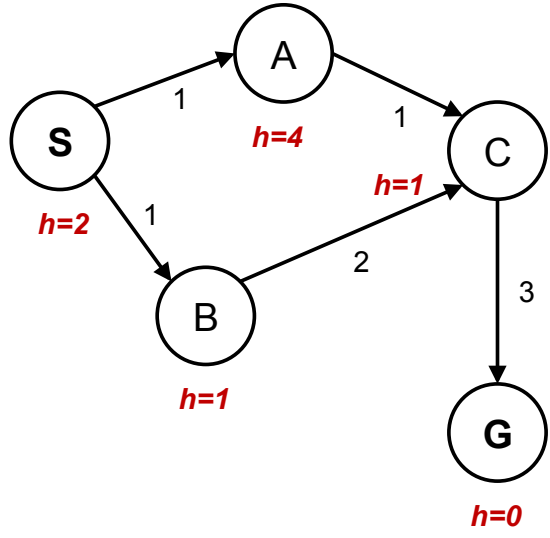
Consider other $h(s)$ fixed.

A: $h(A) = 1$

B: $h(A) = 2$

C: $1 \leq h(A) \leq 2$

D: $0 \leq h(A) \leq 1$



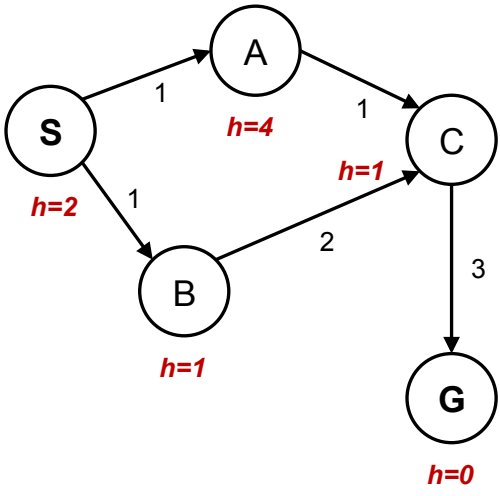
Notes

As it will be explained in the next slides:

$h(A) \leq c(A, C) + h(C) = 2$

$h(S) \leq c(S, A) + h(A)$ it means $h(A) \geq h(S) - c(A, S) = 1$

Consistent heuristics



Admissible h:
$h(A) \leq$ true lowest cost $A \to G$
$h(\text{Goal}) = 0$

Consistent h:
$h(\text{Goal}) = 0$ for all goals
$h(A) - h(C) \leq$ true lowest cost $A \to C$
in general:
$h(p) - h(s) \leq$ true lowest cost $p \to s$ for any pair of parent p and its successor s

$f(n) = g(n) + h(n)$ along a path from start to goal never decreases!

Notes

Our heuristic was admissible.
With *tree search* it would have worked. It would have expanded C and found the alternative, cheaper path.
For graph search, the problem is the $A \to C \to G$ subgraph where the *consistent* heuristic condition is violated.
Based on the graph, there are two constraints for $h(A)$:
$h(S) - h(A) \leq c(S, A)$
$h(A) - h(C) \leq c(A, C)$
Yes, all consistent heuristics are also admissible. Btw., it is not easy to invent a heuristics that is admissible but not consistent.

Summary

- State space graph vs. Search Tree
- Search strategies - properties, complexities
- Evaluating states - `cost_from_start` and `cost_to_go`
- Effectivness – adding heuristic estimates of `cost_to_go`
- Not all heuristics are equally good (admissibility, consistence, informativeness)

Notes

References, further reading

Some figures from [2]. Chapter 2 in [1] provides a compact/dense intro into search algorithms.

[1] Steven M. LaValle.
Planning Algorithms.
Cambridge, 1st edition, 2006.
Online version available at: http://planning.cs.uiuc.edu.

[2] Stuart Russell and Peter Norvig.
Artificial Intelligence: A Modern Approach.
Prentice Hall, 4th edition, 2021.
http://aima.cs.berkeley.edu/.

Notes