HOMEWORK 1
DUE SUNDAY MARCH 29, 2026

Full credit (15 points) can be obtained by adequately answering 7 of the following questions. 5 extra credit points are available for exceptional additional effort. Acknowledge any LLM use.

Exercise 0.1. Draw some pictures/diagrams illustrating some of the ideas from the course so far.

Exercise 0.2. Suppose X is a set and $\mathcal{C} \subseteq \mathcal{P}(X)$ is a nonempty collection of subsets of X. Prove that the following are equivalent:

(1) $\mathcal{C}$ is a boolean algebra of subsets of X,
(2) for every $A, B \in \mathcal{C}$, we have $A \cap B \in \mathcal{C}$ and $X \setminus A \in \mathcal{C}$, and
(3) for every $A, B \in \mathcal{C}$, we have $X \setminus (A \cap B) \in \mathcal{C}$.

Definition. Suppose $X = (X, \tau)$ is a topological space. We say $A \subseteq X$ has the **finite boundary property** if the topological boundary $\operatorname{bd}(A) := \operatorname{cl}(A) \setminus \operatorname{int}(A)$ of A in X is finite. Consider:

$$\mathcal{FBP}(X) \;:=\; \{A \subseteq X : A \text{ has the finite boundary property}\} \;\subseteq\; \mathcal{P}(X)$$

Exercise 0.3. For $A, B \subseteq X$ show the following:

(1) $\operatorname{bd}(A) = \operatorname{bd}(A^c)$
(2) $\operatorname{bd}(A \cup B) \subseteq \operatorname{bd}(A) \cup \operatorname{bd}(B)$
(3) Conclude that $\mathcal{FBP}(X)$ is a boolean algebra of subsets of X.

Exercise 0.4. Now suppose $X = \mathbb{R}$ with the euclidean topology.

(1) Suppose $A = (a, b) \subseteq \mathbb{R}$ is an open interval with $-\infty \leq a < b \leq +\infty$. Show that $A \in \mathcal{FBP}(\mathbb{R})$.
(2) Conclude that $\mathcal{S}_1 \subseteq \mathcal{FBP}(\mathbb{R})$, where $\mathcal{S}_1$ is the boolean algebra of semialgebraic subsets of $\mathbb{R}^1$ (hint: use that $\mathcal{S}_1$ is also the boolean algebra on $\mathbb{R}$ generated by all open intervals).
(3) Show that $\operatorname{bd}(\mathbb{N}) = \mathbb{N}$ is infinite, i.e., $\mathbb{N} \notin \mathcal{FBP}(\mathbb{R})$.
(4) Conclude that $\mathbb{N}$ is not definable in any o-minimal structure on $\mathbb{R}$.

Exercise 0.5. Find, understand, and explain a proof of the fact:

the total real exponential function $\exp : \mathbb{R} \to \mathbb{R}$ *is not a semialgebraic function*

Exercise 0.6. Prove that the complement of a semialgebraic set is semialgebraic.

Exercise 0.7. Prove that a semialgebraic set $X \subseteq \mathbb{R}$ is a finite union of intervals and points.

Exercise 0.8. Let $\Gamma = \Gamma(\mathsf{ReLU}) \subseteq \mathbb{R}^2$ be the graph of the activation function $\mathsf{ReLU} : \mathbb{R} \to \mathbb{R}$.

(1) Find a non-zero polynomial $f \in \mathbb{R}[X, Y]$ such that $\Gamma \subseteq \{f = 0\}$.
(2) Prove there does not exist a polynomial $f \in \mathbb{R}[X, Y]$ such that $\Gamma = \{f = 0\}$.

Exercise 0.9. Suppose $I = (a, b) \subseteq \mathbb{R}$ is an interval with $-\infty \leq a < b \leq +\infty$. Directly construct a semialgebraic bijection $f : (a, b) \to \mathbb{R}$ and explain why your function is semialgebraic.

Exercise 0.10. Consider the semialgebraic set $Q_1 := \{x > 0, y > 0\} \subseteq \mathbb{R}^2$ (the open first quadrant). Prove there does not exist $g(X, Y) \in \mathbb{R}[X, Y]$ such that $Q_1 = \{g(x, y) > 0\}$.

Date: March 9, 2026.

Definition. A **basic closed semialgebraic** subset of $\mathbb{R}^n$ is a set of the form

$$\bigcap_{i=1}^{r}\{x \in \mathbb{R}^n : g_i(x) \geq 0\}$$

where $g_i \in \mathbb{R}[X_1, \ldots, X_n]$ for $i = 1, \ldots, r$.

Exercise 0.11. Show that the semialgebraic set $\{x \geq 0\} \cup \{y \geq 0\} \subseteq \mathbb{R}^2$ is not a basic closed semialgebraic set.

Exercise 0.12. For each $n \geq 0$ define the collection:

$$\mathcal{D}_n \;=\; \{X \subseteq \mathbb{R}^n : X \text{ is a finite union of basic closed semialgebraic sets}\} \;\subseteq\; \mathcal{P}(\mathbb{R}^n)$$

(1) Show that for every $n \geq 0$ and $X \in \mathcal{D}_n$ that X is topologically closed in $\mathbb{R}^n$
(2) Find an example of $n \geq 1$ and $X \in \mathcal{D}_n$ such that $X^c \notin \mathcal{D}_n$.
(3) Find an example of $n \geq 1$ and $X \in \mathcal{D}_{n+1}$ such that $\pi_n[X] \notin \mathcal{D}_n$.
(4) Show that the sequence $\mathcal{D} = (\mathcal{D}_n)_{n\geq 0}$ of collections of sets satisfies Axioms (S2), (S3), (S5), and (S6), but does not satisfy axioms (S1), (S4), (O1), or (O2).

Exercise 0.13. For each $n \geq 0$ show the following equality of collections:

$$\mathcal{D}_n \;=\; \{X \subseteq \mathbb{R}^n : X \text{ is semialgebraic and closed}\}$$

Exercise 0.14. Show we have an equality of structures:

$$\mathbb{R}_{\text{PH}} \;=\; (\mathbb{R}_{\text{alg}}, \sin) \;=\; (\mathbb{R}_{\text{alg}}, f)$$

where $\sin : \mathbb{R} \to \mathbb{R}$ is the total real sine function, and $f : \mathbb{R} \to \mathbb{R}$ is the function:

$$f \;:\; \mathbb{R} \to \mathbb{R}, \quad x \;\mapsto\; f(x) \;:=\; \begin{cases} \frac{1}{n} & \text{if } x \in \mathbb{Q} \text{ and } n \geq 1 \text{ is minimal such that } x = k/n \text{ for some } k \in \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Exercise 0.15. Suppose $\mathcal{R} = (\mathbb{R}; <, \ldots)$ is a structure satisfying (S1)-(S4) and (O1), and suppose $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is a definable set-valued map. Show the following set is definable:

$$\{a \in \mathbb{R}^m : F(a) \text{ is nowhere dense in } \mathbb{R}^n\} \;\subseteq\; \mathbb{R}^m$$

Exercise 0.16. Suppose $\mathcal{R} = (\mathbb{R}; <, +, \cdot, \ldots)$ is a structure satisfying (S1)-(S4),(O1), and (S5)-(S6), and that $I \subseteq \mathbb{R}$ is an interval of the form $(-\infty, a)$ or $(a, +\infty)$ for some $a \in \mathbb{R}$. Show that if $\exp|_I : I \to \mathbb{R}$ is definable in $\mathcal{R}$, then $\exp : \mathbb{R} \to \mathbb{R}$ is definable in $\mathcal{R}$.

Exercise 0.17. For each $-\infty < a < b < +\infty$, show that the restrictions $\exp|_{[a,b]}, \sin|_{[a,b]}, \cos|_{[a,b]}$ are definable in $\mathbb{R}^{\text{RE}}$. Also show the function $\arctan : \mathbb{R} \to \mathbb{R}$ is definable in $\mathbb{R}^{\text{RE}}$.

Exercise 0.18. For each of the following functions $f : \mathbb{R} \to \mathbb{R}$, show that $(\mathbb{R}_{\text{alg}}, f) = \mathbb{R}_{\exp}$.

(1) e^{e^x}
(2) $\tanh(x) = (e^{2x} - 1)/(e^{2x} + 1)$
(3) $\text{sech}(x) = 2e^x/(e^{2x} - 1)$

Furthermore:

(4) Show that the function $f(x) = e^{1/(1+x^2)}$ has the property $(\mathbb{R}_{\text{alg}}, f) \neq \mathbb{R}_{\exp}$.
(5) Give a characterization of a large class of functions f with the property "$(\mathbb{R}_{\text{alg}}, f) = \mathbb{R}_{\exp}$"

Exercise 0.19. Find counterexamples for every implication in the Small Sets Theorem which could fail if we do not assume that $\mathcal{R}$ is o-minimal.