Sequential decisions under uncertainty
Markov Decision Processes (MDP)

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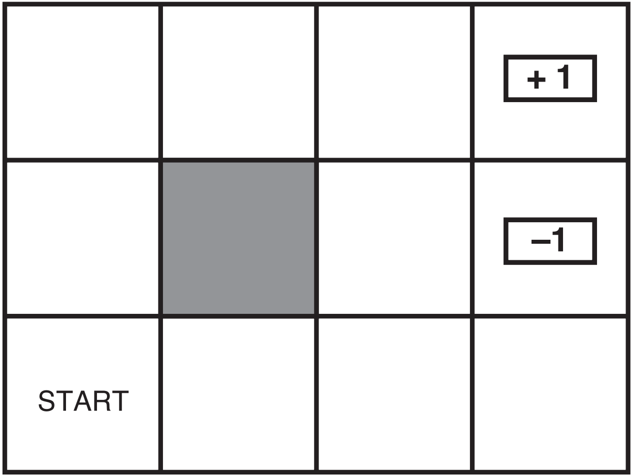
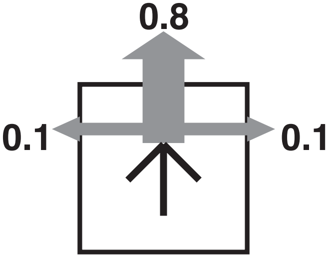
Unreliable actions in *observable* grid world





States $s \in \mathcal{S}$, actions $a \in \mathcal{A}$

(Transition) Model $T(s, a, s') \equiv p(s'|s, a)$ = probability that a in s leads to s'

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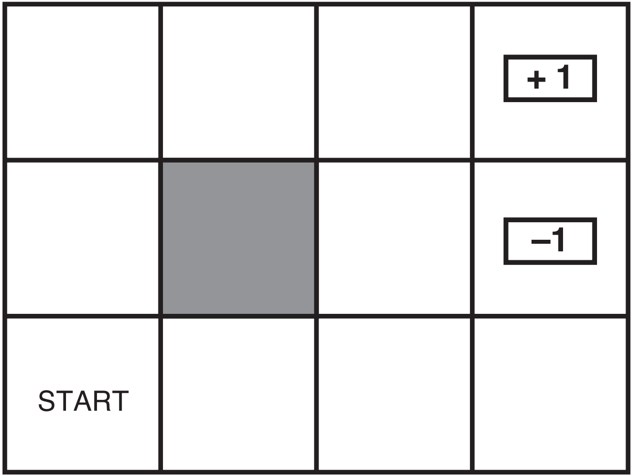

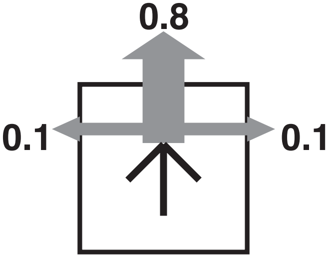

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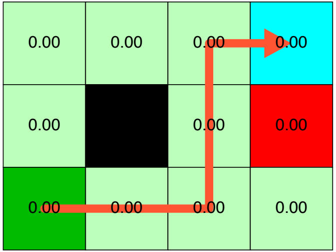
Unreliable (results of) actions

Plan for uncertain world (MDPs)? Policy

- In deterministic world: Plan – sequence of actions from Start to Goal.
- MDPs, we need a policy $\pi : \mathcal{S} \to \mathcal{A}$.
- An action for each possible state. Why *each*?
- What is the *best* policy?



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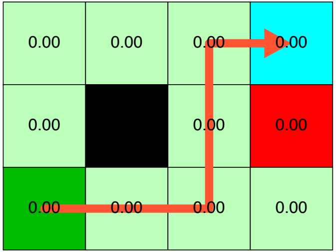

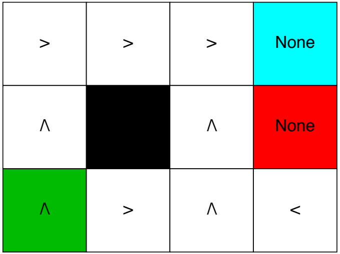

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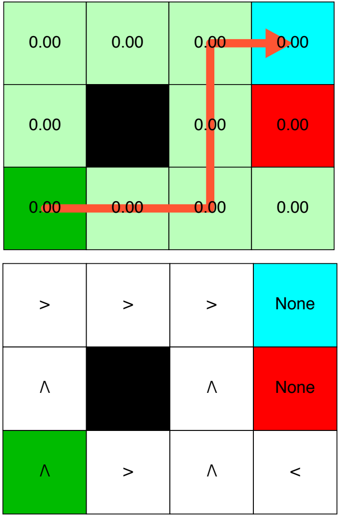

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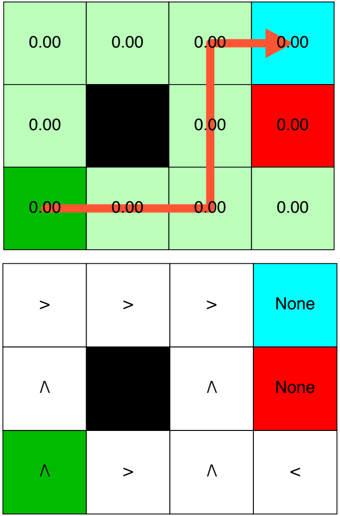


Rewards, Reward function

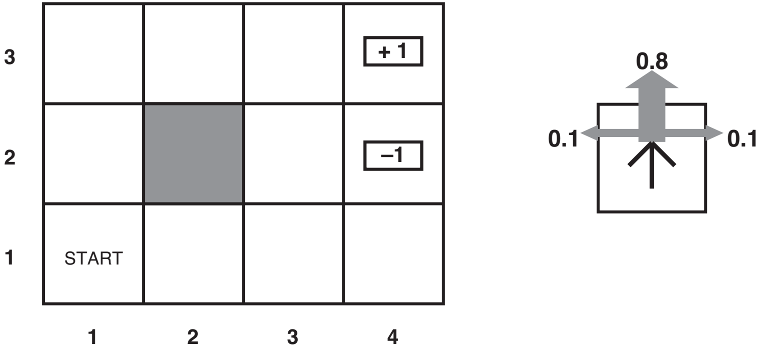
-0.04	-0.04	-0.04	1.00
-0.04		-0.04	-1.00
-0.04	-0.04	-0.04	-0.04

Reward : Robot/Agent takes an action a and it is **immediately** rewarded.

Reward function $r(s)$ (or $r(s,a)$, $r(s,a,s')$)

$$= \begin{cases} -0.04 & \text{(small penalty) for nonterminal states} \\ \pm 1 & \text{for terminal states} \end{cases}$$

Markov Decision Processes (MDPs)



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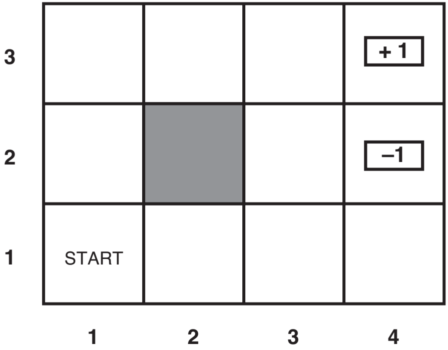

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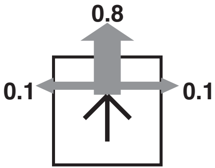

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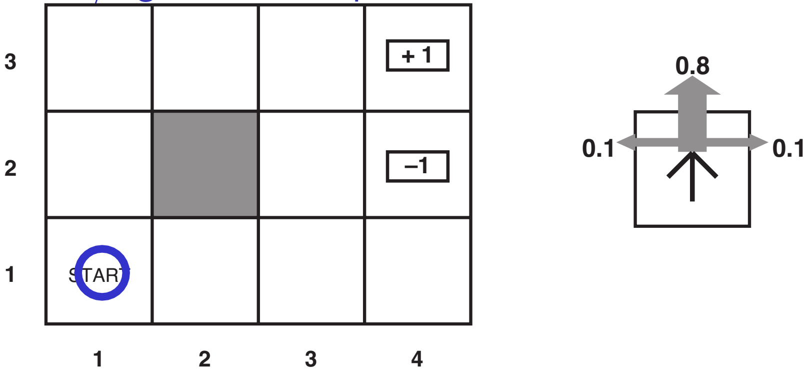
Robot/Agent walk – Episode



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$$S_0, A_0, R_1, S_1, A_1, R_2, S_2, A_2 \ldots$$

Episode : one walk from S_0 to terminal.

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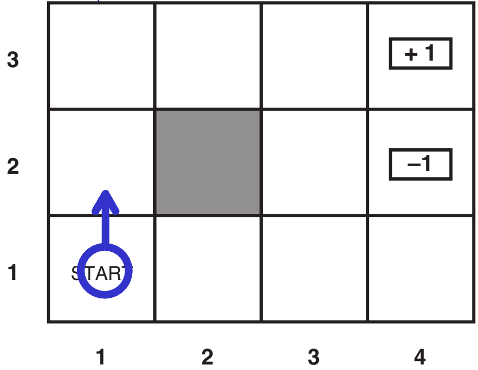

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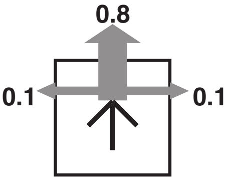

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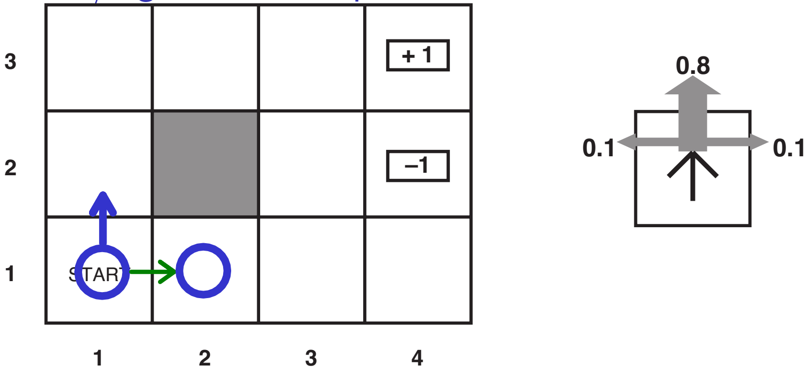

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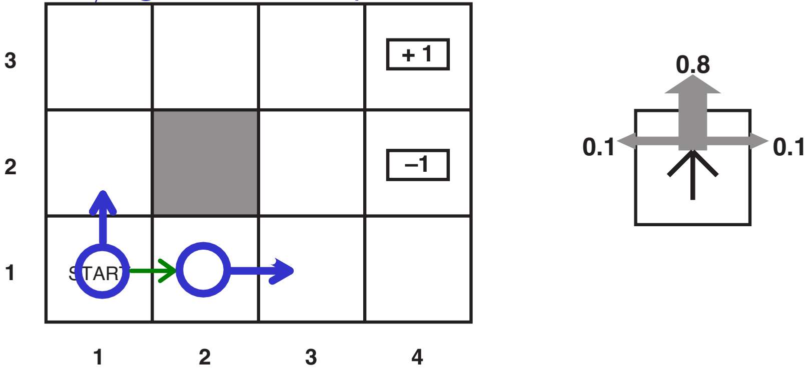

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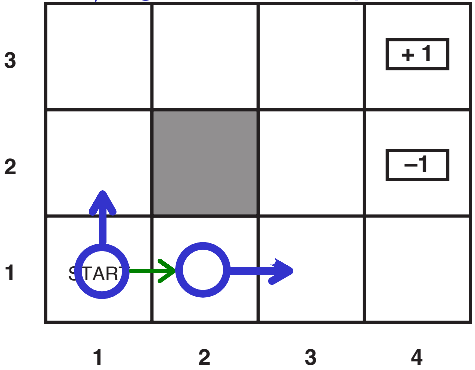

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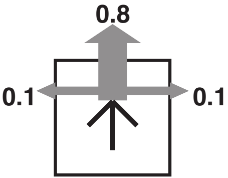

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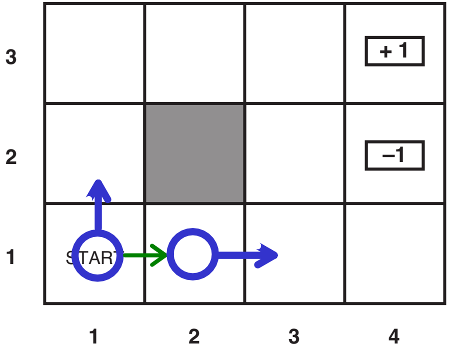
Markovian property

- Given the present state, the future and the past are independent.
- MDP: Markov means action depends only on the current state.
- In search: successor function (transition model) depends on the current state only.



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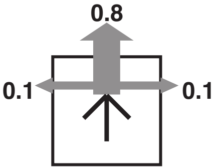

(b)

Desired robot/agent behavior specified through rewards

- Before: shortest/cheapest path
- Solution found by search.
- Environment/problem is defined through the reward function.
- Optimal policy is to be computed/learned.

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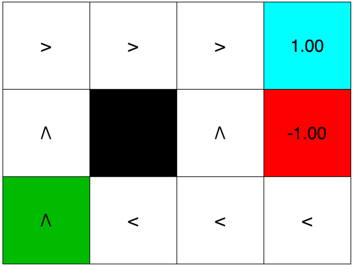
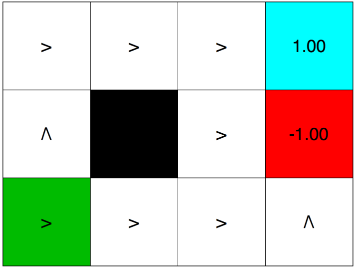
We come back to this in more detail when discussing RL.





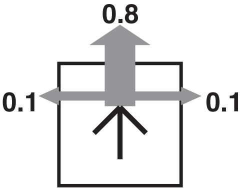
A	B	C
$r(s) \in \{-2, 1, -1\}$	$r(s) \in \{-0.04, 1, -1\}$	$r(s) \in \{-0.01, 1, -1\}$
a	b	c

A: A-a, B-b, C-c

B: A-b, B-a, C-c

C: A-b, B-c, C-a

D: A-c, B-a, C-b



Utilities of sequences; what is a better walk (episode)?

- State reward at time/step t, R_t.
- State at time t, S_t. State sequence $[S_0, S_1, S_2, \ldots,]$

Typically, consider stationary preferences on reward sequences:

$$[R, R_1, R_2, R_3, \ldots] \succ [R, R'_1, R'_2, R'_3, \ldots] \Leftrightarrow [R_1, R_2, R_3, \ldots] \succ [R'_1, R'_2, R'_3, \ldots]$$

If stationary preferences :
Utility (h-history)
$U_h([S_0, S_1, S_2, \ldots,]) = R_1 + R_2 + R_3 + \cdots$

If the horizon is finite - limited number of steps - preferences are nonstationary (depends on how many steps left).

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Agent/Robot wants to live forever

Infinite walks/episodes, additive utilities are infinite.

- How to compare policies ...
- if the sum of rewards is summing forever.

- Assume a terminal state (avoid infinity problem)
- Prefer near future rewards (discount those farther away)

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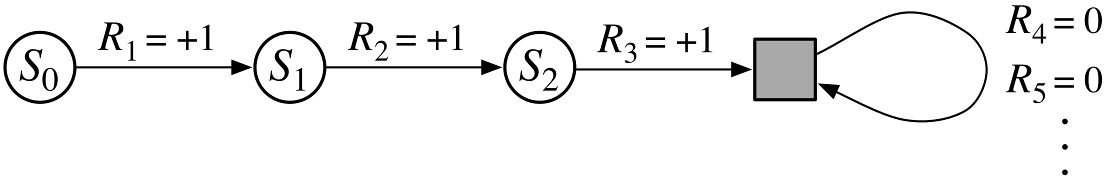
Finite Walk – Episode – and its Return (by assuming a Terminal state)

- Executing policy - sequence of states and **rewards**.
- Episode starts at t, ends at T (ending in a terminal state).
- Return (Utility) of the episode (policy execution)

$$G_t = R_{t+1} + R_{t+2} + R_{t+3} + \cdots + R_T$$



Horizon too far, infinite – Discount rewards

Problem: Infinite lifetime $\Rightarrow$ additive utilities are infinite.

- Finite horizon: termination at a fixed time $\Rightarrow$ nonstationary policy, $\pi(s)$ depends on the time left.
- Absorbing (terminal) state. (sooner or later walk ends here)
- Discounted return , $\gamma < 1$, $R_t \leq R_{\max}$

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \leq \frac{R_{\max}}{1-\gamma}$$

Returns are successive steps related to each other

$$\begin{aligned} G_t &= R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \\ &= R_{t+1} + \gamma (R_{t+2} + \gamma^1 R_{t+3} + \gamma^2 R_{t+4} + \cdots) \\ &= R_{t+1} + \gamma G_{t+1} \end{aligned}$$

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MDPs recap

Markov decision processes (MDPs):

- Set of states $\mathcal{S}$
- Set of actions $\mathcal{A}$
- Transitions $p(s'|s,a)$ or $T(s,a,s')$
- Reward function $r(s,a,s')$; and discount γ
- Alternative to last two: $p(s',r|s,a)$.

MDP quantities:

- (deterministic) Policy $\pi(s)$ – choice of action for each state
- Return (Utility) of an episode (sequence) – sum of (discounted) rewards.

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Expected Return of a policy π

- Executing policy $\pi \rightarrow$ sequence of states (and rewards).
- Utility of a state sequence.
- But actions are unreliable - environment is stochastic.
- Expected return of a policy π.

Starting at time t, i.e. S_t,

$$U^{\pi}(S_t) \doteq \mathrm{E}^{\pi}\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1}\right]$$

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- But actions are unreliable - environment is stochastic.
- Expected return of a policy π.

Starting at time t, i.e. S_t,

$$U^{\pi}(S_t) \doteq \mathrm{E}^{\pi}\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1}\right]$$

(State) Value functions given policy π, Bellman expectation equation

Expected return from that state (state, action)

Value function

$$v^{\pi}(s) \stackrel{def}{=} \mathsf{E}^{\pi}\left[G_t \mid S_t = s\right] = \mathsf{E}^{\pi}\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \,\middle|\, S_t = s\right]$$

Action-value function (q-function)

$$q^{\pi}(s,a) \stackrel{def}{=} \mathsf{E}^{\pi}\left[G_t \mid S_t = s, A_t = a\right] = \mathsf{E}^{\pi}\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \,\middle|\, S_t = s, A_t = a\right]$$

Optimal policy π^*, and optimal value $v^*(s)$

$v^*(s) =$ expected (discounted) sum of rewards (until termination) assuming *optimal* actions.

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Example 1, Robot *deterministic*: $r(s) = \{-0.04, 1, -1\}$, $\gamma = 0.999999$, $\epsilon = 0.03$

	0	1	2	3	
0	0.88	0.92	0.96	1.00	0
1	0.84		0.92	-1.00	1
2	0.80	0.84	0.88	0.84	2
	0	1	2	3	

	0	1	2	3	
0	>	>	>	None	0
1	/\		/\	None	1
2	/\	>	/\	<	2
	0	1	2	3	

Optimal policy π^*, and optimal value $v^*(s)$

$v^*(s)$ = expected (discounted) sum of rewards (until termination) assuming *optimal* actions.

Example 2, Robot *non-deterministic*: $r(s) = \{-0.04, 1, -1\}$, $\gamma = 0.999999$, $\epsilon = 0.03$

	0	1	2	3	
0	0.81	0.87	0.92	1.00	0
1	0.76		0.66	-1.00	1
2	0.71	0.66	0.61	0.39	2
	0	1	2	3	

	0	1	2	3	
0	>	>	>	None	0
1	/\		/\	None	1
2	/\	<	<	<	2
	0	1	2	3	

Optimal policy π^*, and optimal value $v^*(s)$

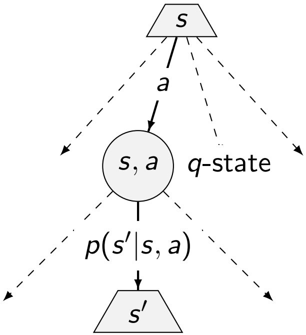
$v^*(s)$ = expected (discounted) sum of rewards (until termination) assuming *optimal* actions.

Example 3, Robot *non-deterministic*: $r(s) = \{-0.01, 1, -1\}$, $\gamma = 0.999999$, $\epsilon = 0.03$

	0	1	2	3	
0	0.95	0.96	0.98	1.00	0
1	0.94		0.89	-1.00	1
2	0.92	0.91	0.90	0.80	2
	0	1	2	3	

	0	1	2	3	
0	>	>	>	1.00	0
1	∧		<	-1.00	1
2	∧	<	<	∨	2
	0	1	2	3	

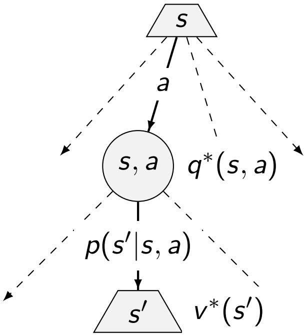
MDP search tree (Expectimax search)



MDP search tree (Expectimax search)

The value of a q-state (s, a):

$$q^*(s, a) = \sum_{s'} p(s'|a, s) \left[r(s, a, s') + \gamma\, v^*(s')\right]$$



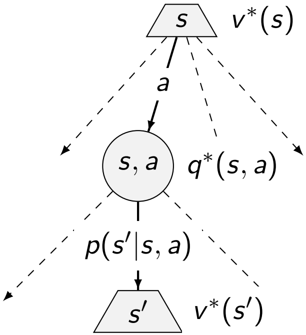
MDP search tree (Expectimax search)

The value of a q-state (s, a):

$$q^*(s,a) = \sum_{s'} p(s'|a,s)\left[r(s,a,s') + \gamma\, v^*(s')\right]$$

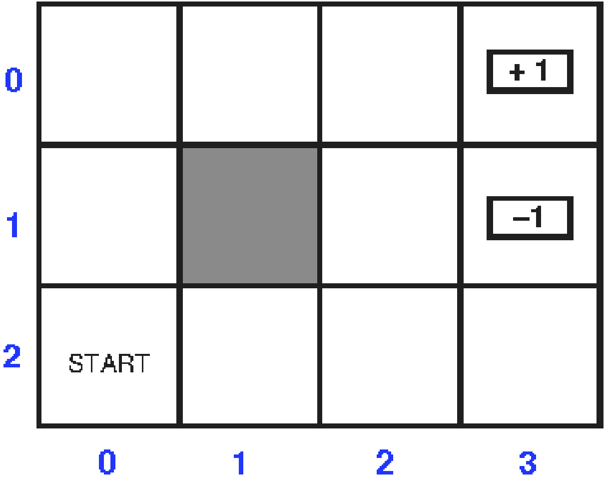
The value of a state s:

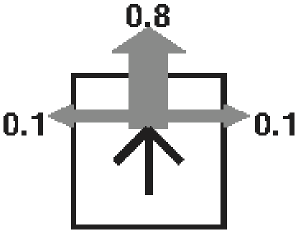
$$v^*(s) = \max_a q^*(s,a)$$



Bellman (optimality) equation

$$v^*(s) = \max_{a \in A(s)} \sum_{s'} p(s'|a,s) \left[r(s,a,s') + \gamma v^*(s') \right]$$





Value iteration – turn Bellman equation into Bellman update

$$v^*(s) = \max_{a \in A(s)} \sum_{s'} p(s'|a,s) \left[r(s,a,s') + \gamma v^*(s') \right]$$

- Start with arbitrary $V_0(s)$ (except for terminals)
- Compute Bellman update (one ply of expectimax from each state)

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} p(s'|s,a) V_k(s')$$

- Repeat until convergence

The idea: Bellman update makes local consistency with the Bellmann equation. Everywhere locally consistent $\Rightarrow$ globally optimal.

Value iteration algorithm is an example of Dynamic Programming method.

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The idea: Bellman update makes local consistency with the Bellmann equation. Everywhere locally consistent $\Rightarrow$ globally optimal.

Value iteration algorithm is an example of Dynamic Programming method.

Value iteration - Complexity of one estimation sweep

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} p(s'|s,a) V_k(s')$$

A: $O(AS)$

B: $O(S^2)$

C: $O(AS^2)$

D: $O(A^2S^2)$

Value iteration demo

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a) V_k(s')$$

	0	1	2	3	
0	0.81	0.87	0.92	1.00	0
1	0.76		0.66	-1.00	1
2	0.71	0.66	0.61	0.39	2
	0	1	2	3	

Convergence

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a) V_k(s')$$

$$\gamma < 1$$

$$-R_{\max} \le R(s) \le R_{\max}$$

Max norm:

$$\|V\|_\infty = \max_s |V(s)|$$

$$U([s_0, s_1, s_2, \ldots, s_\infty]) = \sum_{t=0}^{\infty} \gamma^t R(s_t) \le \frac{R_{\max}}{1-\gamma}$$

Convergence

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a) V_k(s')$$

$$\gamma < 1$$

$$-R_{\max} \leq R(s) \leq R_{\max}$$

Max norm:

$$\|V\|_\infty = \max_s |V(s)|$$

$$U([s_0, s_1, s_2, \ldots, s_\infty]) = \sum_{t=0}^{\infty} \gamma^t R(s_t) \leq \frac{R_{\max}}{1-\gamma}$$

Convergence cont'd

$V_{k+1} \leftarrow BV_k \ldots B$ as the Bellman update $V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} p(s'|s,a) V_k(s')$

$\|BV_k - BV'_k\|_\infty \leq \gamma \|V_k - V'_k\|_\infty$
$\|BV_k - V_{\text{true}}\|_\infty \leq \gamma \|V_k - V_{\text{true}}\|_\infty$
Rewards are bounded, at the beginning then Value error is
$\|V_0 - V_{true}\|_\infty \leq \frac{2R_{\max}}{1-\gamma}$
We run N iterations and reduce the error by factor γ in each and want to stop the error is below ϵ:
$\gamma^N 2R_{\max}/(1-\gamma) \leq \epsilon$ Taking logs, we find: $N \geq \frac{\log(2R_{\max}/\epsilon(1-\gamma))}{\log(1/\gamma)}$
To stop the iteration we want to find a bound relating the error to the size of *one* Bellman update for any given iteration.
If we stop when

$$\|V_{k+1} - V_k\|_\infty \leq \frac{\epsilon(1-\gamma)}{\gamma}$$

then also: $\|V_{k+1} - V_{\text{true}}\|_\infty \leq \epsilon$ Proof on the next slide

Convergence cont'd

$\|V_{k+1} - V_{\text{true}}\|_\infty \leq \epsilon$ is the same as $\|V_{k+1} - V_\infty\|_\infty \leq \epsilon$
Assume $\|V_{k+1} - V_k\|_\infty = \text{err}$
In each of the following iteration steps we reduce the error by the factor γ (because $\|BV_k - V_{\text{true}}\|_\infty \leq \gamma \|V_k - V_{\text{true}}\|_\infty$). Till ∞, the total sum of reduced errors is:

$$\text{total} = \gamma\text{err} + \gamma^2\text{err} + \gamma^3\text{err} + \gamma^4\text{err} + \cdots = \frac{\gamma\text{err}}{(1-\gamma)}$$

We want to have total $< \epsilon$.

$$\frac{\gamma\text{err}}{(1-\gamma)} < \epsilon$$

From it follows that

$$\text{err} < \frac{\epsilon(1-\gamma)}{\gamma}$$

Hence we can stop if $\|V_{k+1} - V_k\|_\infty < \epsilon(1-\gamma)/\gamma$

Value iteration algorithm

function VALUE-ITERATION(env,ϵ) **returns:** state values V
  input: env - MDP problem, ϵ
  $V' \leftarrow 0$ in all states
  repeat ▷ iterate values until convergence
    $V \leftarrow V'$ ▷ keep the last known values (deepcopy)
    $\delta \leftarrow 0$ ▷ reset the max difference
    for each state s **in** S **do**
      $V'[s] \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a)V(s')$
      if $|V'[s] - V[s]| > \delta$ **then** $\delta \leftarrow |V'[s] - V[s]|$
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  until $\delta < \epsilon(1-\gamma)/\gamma$

Sync vs. async Value iteration

function VALUE-ITERATION(env,ϵ) **returns:** state values V

input: env - MDP problem, ϵ

$V' \leftarrow 0$ in all states

repeat ▷ iterate values until convergence

$V = V'$ ▷ don't keep the last known values

$\delta \leftarrow 0$ ▷ reset the max difference

for each state s **in** S **do**

$V'[s] \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a) V(s')$

if $|V'[s] - V[s]| > \delta$ **then** $\delta \leftarrow |V'[s] - V[s]|$

until $\delta < \epsilon(1-\gamma)/\gamma$

What we have learned

- Uncertain outcome of an action
- Optimal policy (strategy, sequence of decisions) maximizes *expected* return (utility, sum of rewards)
- (State) Value function given policy
- Value iteration method - through local (optimal) updated to global optimality

References

Some figures from [1] (chapter 17) but notation slightly changed in order to adapt notation from [2] (chapters 3, 4) which will help us in the Reinforcement Learning part of the course. Note that the book [2] is available on-line.

[1] Stuart Russell and Peter Norvig.
Artificial Intelligence: A Modern Approach.
Prentice Hall, 3rd edition, 2010.
http://aima.cs.berkeley.edu/.

[2] Richard S. Sutton and Andrew G. Barto.
Reinforcement Learning; an Introduction.
MIT Press, 2nd edition, 2018.
http://www.incompleteideas.net/book/the-book-2nd.html.