Quantum Computing 2026 - Exercise Sheet 2

Introduction to Quantum Computing

1 (Basics tensor products). *(a) Compute the following tensor products of states.*

(I) $|0\rangle \otimes |1\rangle$

(II) $|1\rangle \otimes |1\rangle$

(III) $|1\rangle \otimes |0\rangle \otimes |1\rangle$

(IV) $|0\rangle \otimes |1\rangle \otimes |1\rangle$.

(b) Compute the following tensor products of operators.

(I) $\sigma_x \otimes \sigma_y$

(II) $\sigma_z \otimes \sigma_x$

(III) $\mathbb{I} \otimes \sigma_z$

(IV) $\sigma_z \otimes \mathbb{I} \otimes \sigma_x$

$$\text{(a) } (i) \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad (ii) \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \quad (iii) \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{pmatrix}$$

$$(iv) \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\\0\\0\\0\\0\end{pmatrix}$$

$$\text{(b) } (i) \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \otimes \begin{pmatrix}0 & -i\\i & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & -i\\0 & 0 & i & 0\\0 & -i & 0 & 0\\i & 0 & 0 & 0\end{pmatrix}, \quad (ii) \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \otimes \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1 & 0 & 0\\1 & 0 & 0 & 0\\0 & 0 & 0 & -1\\0 & 0 & -1 & 0\end{pmatrix}$$

$$(iii) \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\0 & -1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & -1\end{pmatrix}$$

$$(iv) \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \otimes \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & -1 & 0\\0 & 0 & 0 & -1\end{pmatrix} \otimes \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & -1 & 0 & 0\\0 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & -1\\0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\end{pmatrix}$$

2 (Entanglement). *(a) Are these states entangled:*

(I) $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$

(II) $|\psi_2\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$

(III) $|\psi_3\rangle = \frac{1}{\sqrt{2}}(|001\rangle + |011\rangle)$.

(b) Apply the following multipartite measurements

(I) $\sigma_x \otimes \sigma_z |\psi_1\rangle$ (II) $\sigma_y \otimes \mathbb{I} \otimes \sigma_x |\psi_2\rangle$ (III) $\mathbb{I} \otimes \sigma_z \otimes \sigma_z |\psi_3\rangle$

(a) (i) It is entangled, as it cannot be written as the tensor product of the individual qubits.
(ii) It is entangled, as it cannot be written as the tensor product of the individual qubits.
(iii) It is not entangled, as we can write

$$|\psi_3\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle$$

(b) For the entangled states we have to compute the full system as we cannot act on each qubit separately. We see how for the separable state the computation is much easier. (i) $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

(ii) $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ -i \\ 0 \\ 0 \\ i \\ 0 \\ 0 \\ i \end{pmatrix} = \frac{i}{\sqrt{3}}(-|001\rangle + |101\rangle + |111\rangle)$

$(iii)\mathbb{I}\,|0\rangle \otimes \frac{1}{\sqrt{2}}\sigma_z(|0\rangle + |1\rangle) \otimes \sigma_z\,|1\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes (-|1\rangle) = \frac{1}{\sqrt{2}}(-|001\rangle + |011\rangle)$.

3 (Single-qubit gates)**.** *Build the gates that do the following transformations:*

(a) $|0\rangle \to |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |1\rangle \to |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

(b) $|+\rangle \to |+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle), \quad |-\rangle \to |-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$

(c) $|0\rangle \to |+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle), \quad |1\rangle \to |-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$

(a) We see that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ so by inspection we build the transformation matrix $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ This is the Hadamard gate, H.
(b) Finally, by the same method we have $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$ and by inspection we find $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, the phase gate, S.
(c) Similarly $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$ and by inspection we find $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$. This is equivalent to applying the Hadamard gate and the phase gate afterwards.

4 (Multi-qubit gate)**.** *Build the gate that implements the following transformations*

$$|00\rangle \to |00\rangle\,, \quad |01\rangle \to |01\rangle\,, \quad |10\rangle \to |11\rangle\,, \quad |11\rangle \to |10\rangle\,.$$

By looking at the linear map that satisfies the following operations:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

we find the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

which is the CNOT gate. This gate has an control and a target qubits, if the control qubit (the first qubit in this case) is 1 then we change the state of the second qubit, if it is 0 then we leave the second qubit as it is.

5 (Creation of Entanglement)**.** *Build the entangled state (Bell state)* $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ *starting with the state* $|00\rangle$*, by using the gates H and CNOT. Draw the circuit implementing this.*

If we apply the CNOT gate right away we will not do anything to the qubits as both are one. However if we apply the H gate first, say to the first qubit, then we get

$$(H \otimes \mathbb{I}) |00\rangle = \frac{1}{\sqrt{2}} |+\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \otimes |0\rangle .$$

Now we can let the first qubit be the control qubit for the CNOT gate, such that if it is 1 then we change the second qubit. So we get

$$\text{CNOT}(H \otimes \mathbb{I}) |00\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle). \tag{1}$$

With this we have created an entangled state from 0.

6 (Teleportation protocol)**.** *Alice and Bob share a state* $|\beta_{00}\rangle$ *initialized as* $|00\rangle$*. They go to separated places and Alice wants to teleport an unknown state* $|\phi\rangle = \alpha |0\rangle + \beta |1\rangle$*. By teleporting here we mean sending the state to Bob only with classical information. The full system's initial state is then* $|\psi_0\rangle = |\phi\rangle |\beta_{00}\rangle$*.*

(a) Entangle the shared state $|\beta_{00}\rangle$ *and write the new full state* $|\psi_1\rangle$*.*

(b) We now want to correlate Alice's unknown state with the shared state. For this apply a CNOT gate with Alice's unknown state as control and Alice's shared state as the target. Then apply a Hadamard gate to Alice's unknown state. What is the full state $|\psi_2\rangle$*.*

(c) Rearrange $|\psi_2\rangle$ *in the following form*

$$|\psi_3\rangle = \frac{1}{2} (|00\rangle |a_1\rangle + |01\rangle |a_2\rangle + |10\rangle |a_3\rangle + |11\rangle |a_4\rangle).$$

Find Bob's states $|a_1\rangle, |a_2\rangle, |a_3\rangle, |a_4\rangle$*.*

(d) Alice measures her two qubits with results $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ *and sends these 2 bits of information to Bob. If Alice measures* $|00\rangle$ *then Bob will have the original state* $|\phi\rangle$*. What measurements does Bob have to apply to his state to recover* $|\phi\rangle$ *if Alice sends the bits* $01, 10, 11$*?*

(e) Draw the quantum circuit for the full teleportation protocol.

(a) From the previous exercise we can entangle $|\beta_{00}\rangle$ *with a CNOT and H gates, so we can write*

$$|\psi_1\rangle = |\phi\rangle (CNOT(H \otimes \mathbb{I})) |\beta_{00}\rangle = |\phi\rangle |\Phi^+\rangle = \frac{1}{\sqrt{2}} \Big(\alpha |0\rangle (|00\rangle + |11\rangle) + \beta |1\rangle (|00\rangle + |11\rangle)\Big) \tag{2}$$

(b) We start by applying a CNOT gate with Alice's unknown state as a control and her shared state as target:

$$CNOT_{1,2} |\phi\rangle |\Phi^+\rangle = \frac{1}{\sqrt{2}} \Big(\alpha |0\rangle (|00\rangle + |11\rangle) + \beta |1\rangle (|10\rangle + |01\rangle)\Big)$$

We see how only Alice's shared qubit related to the 1 in the unknown qubit are affected. Now we apply the Hadamard gate to Alice's unknown state to obtain

$$\begin{aligned} |\psi_2\rangle &= (H \otimes \mathbb{I} \otimes \mathbb{I}) \frac{1}{\sqrt{2}} \Big(\alpha |0\rangle (|00\rangle + |11\rangle) + \beta |1\rangle (|10\rangle + |01\rangle)\Big) \\ &= \frac{1}{2} \Big(\alpha(|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + \beta(|0\rangle - |1\rangle)(|10\rangle + |01\rangle)\Big). \end{aligned}$$

(c) We can rewrite the previous state in (b) as

$$|\psi_3\rangle = \frac{1}{2} (|00\rangle (\alpha |0\rangle + \beta |1\rangle) + |01\rangle (\alpha |1\rangle + \beta |0\rangle) + |10\rangle (\alpha |0\rangle - \beta |1\rangle) + |11\rangle (\alpha |1\rangle - \beta |0\rangle)).$$

(d) Our goal is for Bob to recover the state $|\phi\rangle = \alpha |0\rangle + \beta |1\rangle$ *so, if he receives the bits 01, he will have to transform*

$$\alpha |1\rangle + \beta |0\rangle \to \alpha |0\rangle + \beta |1\rangle$$

This can be achieved by switching 0 and 1, so applying an X gate.
If Bob receives the bits 10, he will have to transform

$$\alpha |0\rangle - \beta |1\rangle \to \alpha |0\rangle + \beta |1\rangle$$

This can be achieved by multiplying the qubit 1 with a factor of -1, so applying an Z gate.
Finally, if Bob receives the bits 11, he will have to transform

$$\alpha |1\rangle - \beta |0\rangle \rightarrow \alpha |0\rangle + \beta |1\rangle$$

This can be achieved by first switching 0 and 1 and then multiplying the qubit 1 with a factor of -1, so applying an X and Z gates consecutively.
(e) It should look something like this