Quantum Computing 2025 - Exercise Sheet 1

Basics of Quantum Mechanics

1. *Given the orthonormal basis states* $\left\{ |u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

a) Show that the 'in' and 'out' states defined as:

$$|i\rangle = \frac{1}{\sqrt{2}}(|u\rangle + i|d\rangle)$$

$$|o\rangle = \frac{1}{\sqrt{2}}(|u\rangle - i|d\rangle)$$

are orthonormal.

b) Show that the 'left' and 'right' states defined as:

$$|l\rangle = \frac{1}{\sqrt{2}}(|u\rangle + |d\rangle)$$

$$|r\rangle = \frac{1}{\sqrt{2}}(|u\rangle - |d\rangle)$$

are orthonormal.

c) Show that the expressions for calculating the expectation values of an operator, $\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle = \sum_n a_n P(|a_n\rangle)$ *where* $a_n, |a_n\rangle$ *are the eigenvalues and eigenvectors respectively.*

d) Calculate the expectation values of σ_y *in the states* $|u\rangle$ *and* $|i\rangle$*, and of* σ_z *in the state* $|o\rangle$, $|l\rangle$.

a) We take their product, which in the braket notation reads as $\langle o|i\rangle$, and verify that it is 0:

$$\langle o|i\rangle = \frac{1}{\sqrt{2}}(\langle u| + i\langle d|) \cdot \frac{1}{\sqrt{2}}(|u\rangle + i|d\rangle) = \frac{1}{2}\left(\langle u|u\rangle + i\langle u|d\rangle + i\langle d|u\rangle - \langle d|d\rangle\right)$$

Now, recalling $|u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we have:

$$\langle u|u\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \quad , \quad \langle u|d\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\langle d|u\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \quad , \quad \langle d|d\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$$

Then show that they are normal $\langle l|l\rangle = 1, \langle r|r\rangle = 1$ by a similar way as above.

b) Similarly to the above we repeat the same process. For orthogonality show this as

$$\langle l\,|r\rangle = \frac{1}{\sqrt{2}}\left(\langle u| + \langle d|\right) \cdot \frac{1}{\sqrt{2}}\left(|u\rangle - |d\rangle\right) = \frac{1}{2}(\langle u|u\rangle - \langle u|d\rangle + \langle d|u\rangle - \langle dd\rangle) = 0$$

Then show that they are normal $\langle l|l\rangle = 1, \langle r|r\rangle = 1$ by a similar way as above.

c) Here we start with the expression on the RHS and show that it is equivalent to the LHS

$$\langle \hat{A} \rangle = \sum_n a_n P(|a_n\rangle) = \sum_n a_n |\langle a_n|\,\psi\rangle|^2 = \sum_n a_n \langle a_n|\psi\rangle\langle\psi\,|a_n\rangle$$

Note that $\langle a_n|\psi\rangle$ and its conjugate are simply just scalars so we can move them around anywhere we like. This leads to

$$\langle\psi|\,(\sum_n a_n|a_n\rangle\langle a_n|)\,|\psi\rangle$$

The expression in the bracket is just the operator $\hat{A}$ expressed in its eigenbasis. Hence

$$\langle \hat{A} \rangle = \langle\psi|\,\hat{A}\,|\psi\rangle\,.$$

d) We insert the Pauli matrices into the expression for the expectation value of a general operator: $\langle\psi|A|\psi\rangle$

$$\langle u|\sigma_y|u\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ i \end{pmatrix} = 0$$

$$\langle l|\sigma_y|l\rangle = \frac{1}{2}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix} = 1$$

Alternatively, rather than working with matrix multiplication you can decompose the state into the eigenstates of the desired operator. For example

$$\langle o|\sigma_z|o\rangle = \frac{1}{2}(\langle u| + i\langle d|)\sigma_z(|u\rangle - i|d\rangle)$$

Since we know that $\sigma_z|u\rangle = |u\rangle$ and $\sigma_z|d\rangle = -|d\rangle$ we have

$$\langle o|\sigma_z|o\rangle = \frac{1}{2}(\langle u| + i\langle d|)(|u\rangle + i|d\rangle) = \frac{1}{2}(1-1) = 0$$

Similarly

$$\langle l|\,\sigma_z\,|l\rangle = \frac{1}{2}(\langle u| + \langle d|)\sigma_z(|u\rangle + |d\rangle) = \frac{1}{2}(\langle u| + \langle d|)(|u\rangle - |d\rangle) = \frac{1}{2}(1-1) = 0$$

2. *a) Normalise the state*

$$|\psi\rangle = 3i|u\rangle + (1-2i)|d\rangle.$$

b) For this (normalised) state, calculate the probability of getting both positive $(+1)$ *and negative* (-1) *spin eigenvalues by measuring* σ_z.

a) Normalization means that taking the norm of the state ,$|\psi\rangle$, is unity i.e. $\sqrt{\langle\psi|\psi\rangle} = 1$.
In this case however, we have:

$$\langle\psi|\psi\rangle = (-3i)(3i) + (1+2i)(1-2i) = 9 + 5 = 14 \neq 1.$$

We should then rescale our state by a constant N, $|\psi\rangle \Rightarrow N\cdot|\psi\rangle$ such that the result is 1:

$$\langle N\psi|N\psi\rangle = N^2\overbrace{\langle\psi|\psi\rangle}^{=14} = 1 \Rightarrow N = \frac{1}{\sqrt{14}}$$

So our new normalised state is

$$|\psi\rangle = \frac{3i}{\sqrt{14}}|u\rangle + \frac{(1-2i)}{\sqrt{14}}|d\rangle.$$

b) $P_\psi(+) = |\langle u|\psi\rangle|^2 = |\frac{3i}{\sqrt{14}}|^2 = \frac{9}{14}$. For $P_\psi(-)$, we know that this is the only other possible outcome so $P_\psi(-) = 1 - \frac{9}{14} = \frac{5}{14}$.

3. *(Operators and Measurements) Find the eigenvalues, eigenvectors and diagonal representations for the following operators:*

a) $\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ *b)* $\sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$ *c)* $\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

(d) Write the above in matrices in terms of Dirac notation: i) in the basis $\{|u\rangle, |d\rangle\}$ *and ii) in the basis of their eigenvectors.*

(e) For the wavefunction $|\psi\rangle = \sqrt{\frac{2}{3}}|u\rangle + \sqrt{\frac{1}{3}}|d\rangle$, *calculate the expectation value for* σ_y

(f) Show that for each of the sets of orthonormal (show this if you like) eigenvectors the completeness relation

$$\mathbb{I}_n = \sum_i |\psi_i\rangle\langle\psi_i|$$

(g) A quantum state can also be written as $|\psi\rangle = \cos(\frac{\theta}{2})\,|u\rangle + e^{i\varphi}\sin(\frac{\theta}{2})\,|d\rangle$, *where the angles can be seen in the Bloch sphere*

Draw each of the eigenvectors and the state $|\psi\rangle = \sqrt{\frac{3}{4}}|u\rangle - i\sqrt{\frac{1}{4}}|d\rangle$ *on the Bloch sphere.*

a) The eigenvalues of σ_x are determined by calculating the characteristic polynomial

$$\det|\sigma_x - \lambda I| = \det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = 0$$

therefore $\lambda_\pm = \pm 1$.
For each of the eigenvalues, we find the eigenvectors by solving the following system of equations

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_\pm \\ b_\pm \end{pmatrix} = \lambda_\pm \begin{pmatrix} a_\pm \\ b_\pm \end{pmatrix}$$

For λ_+:

$$b_+ = a_+ = 1,$$

The choice of 1 here is arbitrary,as we normalize the vector by a factor of $\sqrt{|a|^2 + |b|^2}$ hence the corresposning vector is

$$v_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For λ_-:

$$b_- = -a_- = -1$$

and normalizing in the way we did before we have

$$v_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

You should recognize these as the $|l\rangle = \frac{1}{\sqrt{2}}(|u\rangle) + |d\rangle$ and $|r\rangle = \frac{1}{\sqrt{2}}(|u\rangle) - |d\rangle$.
For the next two, the procedure above is the same and we obtain:
b) eigenvalues $\lambda_\pm = \pm 1$ with eigenvectors

$$v_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$v_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

these are the in and out states, $|i\rangle = \frac{1}{\sqrt{2}}(|u\rangle + i\,|d\rangle)$ and $|o\rangle = \frac{1}{\sqrt{2}}(|u\rangle - i\,|d\rangle)$.

c) Again, the values are $\lambda_\pm = \pm 1$ and the with vectors

$$v_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$v_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which are clearly just the $|u\rangle$ and $|d\rangle$.
d) i) Any operator $O = \sum_i \lambda_{ij} |i\rangle\langle j|$, by applying the matrices to $|u\rangle, |d\rangle$, it can be seen that:

$$\sigma_x = |u\rangle\langle d| - |d\rangle\langle u|$$
$$\sigma_y = -i\,|u\rangle\langle d| + i\,|d\rangle\langle u|$$
$$\sigma_z = |u\rangle\langle u| - |d\rangle\langle d|$$

ii) Another way to write operators in Dirac notation is as the linear combination of outer products is as $O = \sum_i \lambda_i |i\rangle\langle i|$ where here λ_i are the eigenvalues and $|i\rangle$ are the eigenvectors. This is possible since, all operators are diagonalizable in their own basis, therefore:

$$\sigma_x = |l\rangle\langle l| - |r\rangle\langle r|$$
$$\sigma_y = |i\rangle\langle i| - |o\rangle\langle o|$$
$$\sigma_z = |u\rangle\langle u| - |d\rangle\langle d|$$

e) There are many ways this computation could be performed, the two most commons ways are by i) Matrix representation and ii) Dirac notation i)

$$\langle \sigma_y \rangle = \langle \psi | \sigma_y | \psi \rangle = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \end{pmatrix} \begin{pmatrix} -i\sqrt{\frac{1}{3}} \\ +i\sqrt{\frac{2}{3}} \end{pmatrix} = \frac{-2i}{3} + \frac{2i}{3} = 0$$

ii)

$$\langle \sigma_y \rangle = \langle \psi | \sigma_y | \psi \rangle = (\sqrt{\frac{2}{3}}\langle u| + \sqrt{\frac{1}{3}}\langle d|)(-i\,|u\rangle\langle d| + i\,|d\rangle\langle u|)(\sqrt{\frac{2}{3}}|u\rangle + \sqrt{\frac{1}{3}}|d\rangle) = (\sqrt{\frac{2}{3}}\langle u| + \sqrt{\frac{1}{3}}\langle d|)(-i\sqrt{\frac{1}{3}}|u\rangle + i\sqrt{\frac{2}{3}}|d\rangle) = 0$$

f) Here we will just show this for the Z-basis ($\{|u\rangle, |d\rangle\}$), but the same principle applies for the others

$$\sum_i |\psi_i\rangle\langle\psi_i| = |u\rangle\langle u| + |d\rangle\langle d| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

g)
For the eigenvectors of σ_z we have $\{|u\rangle, |d\rangle\}$ by comparing with the general formula for a wavefuction written in spherical coordinates (θ, ϕ) $|\psi\rangle = \cos(\theta/2)|u\rangle + e^{i\phi}\sin(\theta/2)|d\rangle$

we see that for $|u\rangle$:

$$\cos(\theta/2) = 1 \rightarrow \theta/2 = 0 \rightarrow \theta = 0$$

and ϕ is arbitrary.
Continuing in the same way we see

$$|d\rangle \rightarrow \theta = \pi, \phi \text{ is arbitrary}$$

$$|l\rangle \rightarrow \theta = \pi/2, \phi = 0$$

$$|r\rangle \rightarrow \theta = \pi/2, \phi = \pi$$

$$|i\rangle \rightarrow \theta = \pi/2, \phi = \pi/2$$

$$|o\rangle \rightarrow \theta = \pi/2, \phi = -\pi/2$$

for the state $|\psi\rangle = \sqrt{\frac{3}{4}}|u\rangle - i\sqrt{\frac{1}{4}}|d\rangle$:

$$\cos(\theta/2) = \sqrt{\frac{3}{4}} \rightarrow \theta = \frac{\pi}{6}$$
$$e^{i\phi} = -i \rightarrow \phi = \frac{3}{4}\pi$$

4. *(Properties of Hermitian and Unitary Matrices) For a hermitian matrix* $\mathbf{A}$*, that is, a matrix that satisfies* $\mathbf{A} = \mathbf{A}^\dagger$*, show that:*

(a) Different eigenvalues have orthogonal eigenvectors.

(b) All its eigenvalues are real. Does the converse also hold, that is, if the spectrum (the set of all eigenvalues) of a matrix is in $\mathbb{R}$*, is it then a hermitian matrix?*

Now, consider a unitary matrix, one for which

$$UU^\dagger = \mathbb{I} \iff U^\dagger U = \mathbb{I} \iff U^{-1} = U^\dagger$$

(c) Prove that its eigenvalues are of the form $e^{i\theta}$

(d) Prove that eigenvectors of different eigenvalues must be orthogonal

(e) *Now consider two operators* $\mathbf{A}, \mathbf{B}$. *Show that if they are simultaneously diagonalizable, then* $[\mathbf{A}, \mathbf{B}] = 0$

(f) *For any observables* $\mathbf{A}$ *and* $\mathbf{B}$, *and state* $|\psi\rangle$, *derive Heisenberg's uncertainty relation:* $\Delta\mathbf{A}\cdot\Delta\mathbf{B} \geq \frac{1}{2}|\langle\psi|[A,B]|\psi\rangle|$, *where* $(\Delta\mathbf{A})^2 = \sum_a (a - \langle\mathbf{A}\rangle)^2 P(a)$, *is the standard deviation of the operator* $\mathbf{A}$. *To do this, first show that:*

(I) $(\Delta\mathbf{A})^2 = \langle\bar{\mathbf{A}}^2\rangle$ *where* $\bar{\mathbf{A}} = \mathbf{A} - \langle\mathbf{A}\rangle$

(II) $[\mathbf{A}, \mathbf{B}] = [\bar{\mathbf{A}}, \bar{\mathbf{B}}]$

(III) *Now, using these and the Cauchy Schwartz inequality,* $2|X||Y| \geq |\langle X|Y\rangle + \langle Y|X\rangle|$, *derive the uncertainty principle. and defining the states* $|X\rangle = \bar{\mathbf{A}}|\Psi\rangle$, $|Y\rangle = i\bar{\mathbf{B}}|\Psi\rangle$

(g) *Calculate the commutation relations between different combinations of the Pauli-matrices* $\sigma_x, \sigma_y, \sigma_z$. *(Bonus: Can you derive some formula for all possible combination?)*

(h) *Can* σ_x *and* σ_y *be simultaneously observed?*

(i) *Consider a one-dimensional quantum particle in the position representation. The position operator acts as*

$$\hat{x}\psi(x) = x\,\psi(x),$$

and the momentum operator acts as

$$\hat{p}\psi(x) = -i\hbar\frac{d}{dx}\psi(x).$$

Compute the commutator $[\hat{x}, \hat{p}]$.

a) Consider two eigenvectors of A

$$A|\psi\rangle = \lambda_\psi|\psi\rangle, A|\psi\rangle = \lambda_\phi|\phi\rangle$$

Now consider:

$$\langle\psi|A|\phi\rangle = \lambda_\phi\langle\psi|\phi\rangle$$

But also since $A = A^\dagger$ this can be written as:

$$\langle\psi|A^\dagger|\phi\rangle = \lambda_\psi^*\langle\psi|\phi\rangle = \lambda_\psi\langle\psi|\phi\rangle$$

Since eigenvlaues of Hermitian matrices are real as we will show below. These two expressions are equivalent so subtracting one from the other leads to:

$$(\lambda_\phi - \lambda_\psi)\langle\psi|\phi\rangle = 0$$

$\lambda_\psi \neq \lambda_\phi$ so there difference is $\neq 0$. Therefore the eigenvectors must be orthogonal for the above to be true.
b) Consider the matrix element of the adjoint of the operator: $\langle\phi|A^\dagger|\psi\rangle$
The operator can either act on the ket (that is, from the left) or on the bra, in which case it is 'daggered':

$$\langle\phi|A^\dagger|\psi\rangle = \langle A\phi|\psi\rangle$$

Now, we know that for any bra(c)ket we have: $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$
A fact that comes from the very definition of the inner product which is linear in the second argument and anti-linear in the first, see for instance Nielsen and Chuang eqs. (2.13) and (2.15)
Taking this remark into account:

$$\langle\phi|A^\dagger|\psi\rangle = \langle A\phi|\psi\rangle = \langle\psi|A\phi\rangle^* = \langle\psi|A|\phi\rangle^*$$

Particularising for the case where $\phi = \psi$ and taking into account the eigenvalue equation, $\mathbf{A}|\psi\rangle = a|\psi\rangle$, we retrieve the eigenvalues of the operator:

$$\langle\psi|\mathbf{A}|\psi\rangle = \langle\psi|a|\psi\rangle = a\cdot\langle\psi|\psi\rangle = a$$

Last, by assumption, we have $A^\dagger = A$, so:

$$\langle\psi|A^\dagger|\psi\rangle = \langle\psi|A|\psi\rangle = \langle\psi|A|\psi\rangle^* \Rightarrow a = a^*$$

The converse is not true in general, e.g. $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$

c) We start by writing the eigenvalue equation for the unitatry operator U :

$$U|x\rangle = \lambda_x |x\rangle \leftrightarrow \langle x| U^\dagger = \langle x| \lambda_x^*$$

We have normalised our eigenvector, so:

$$\langle x|x\rangle = 1 = \langle x|U^\dagger U|x\rangle = \langle x|\lambda_x^* \lambda_x|x\rangle = |\lambda_x|^2 \langle x|x\rangle = |\lambda_x|^2 \Rightarrow |\lambda_x|^2 = 1 \Rightarrow \lambda_x = e^{i\theta}$$

d) Now, as in the previous example, consider two eigenvector of U :

$$U|x\rangle = \lambda_x |x\rangle \ , \ U|y\rangle = \lambda_y |y\rangle$$

If $\lambda_x \neq \lambda_y$, then:

$$\langle x|y\rangle = \langle x|U^\dagger U|y\rangle = \lambda_x^* \lambda_y \langle x|y\rangle \Rightarrow \langle x|y\rangle (1 - \lambda_x^* \lambda_y) = 0 \overset{\cdot \lambda_x}{\Rightarrow} \langle x|y\rangle (\lambda_x - \lambda_y) = 0$$

e) Since they are diagonal in the same basis, they commute:

$$\mathbf{AB} = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{pmatrix} = \begin{pmatrix} a_1 \cdot b_1 & & \\ & \ddots & \\ & & a_n \cdot b_n \end{pmatrix} = \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{pmatrix} \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = \mathbf{BA}$$

f) Following the reasoning in Susskind 5.4 $\rightarrow$ 5.7, we first prove that $(\Delta \mathbf{A}) = \langle \bar{\mathbf{A}}^2 \rangle$:

$$(\Delta \mathbf{A})^2 = \sum_a (a - \langle \mathbf{A} \rangle)^2 P(a) = \sum_a (a - \langle \mathbf{A} \rangle)^2 |\langle a|\psi\rangle|^2 = \sum_a (a - \langle \mathbf{A} \rangle)^2 \langle a|\psi\rangle^* \langle a|\psi\rangle = \sum_a (a - \langle \mathbf{A} \rangle)^2 \langle \psi|a\rangle \langle a|\psi\rangle =$$

$$= \langle \psi| \underbrace{\sum_a (a - \langle \mathbf{A} \rangle)^2 |a\rangle\langle a|}_{(\mathbf{A} - \langle \mathbf{A} \rangle)^2} \psi\rangle = \langle \bar{\mathbf{A}}^2 \rangle$$

Where the last claim in the brace can be shown using the completeness relation: $\mathbf{A} = \sum_a a|a\rangle\langle a|$:

$$(\mathbf{A} - \langle \mathbf{A} \rangle)^2 = (\sum_a a - \langle \mathbf{A} \rangle |a\rangle\langle a|)(\sum_\alpha \alpha - \langle \mathbf{A} \rangle |\alpha\rangle\langle \alpha|) = \sum_{a,b} (a - \langle \mathbf{A} \rangle)(\alpha - \langle \mathbf{A} \rangle)|a\rangle \underbrace{\langle a|\alpha\rangle}_{\delta_{a\alpha}} \langle \alpha| = \sum_a (a - \langle \mathbf{A} \rangle)^2 |a\rangle\langle a|$$

Secondly, we prove $[\bar{\mathbf{A}}, \bar{\mathbf{B}}] = [\mathbf{A}, \mathbf{B}]$, by computing explicitly the commutator:

$$[\bar{\mathbf{A}}, \bar{\mathbf{B}}] = (\mathbf{A} - \langle \mathbf{A} \rangle)(\mathbf{B} - \langle \mathbf{B} \rangle) - (\mathbf{B} - \langle \mathbf{B} \rangle)(\mathbf{A} - \langle \mathbf{A} \rangle) = \mathbf{AB} - \mathbf{A}\langle \mathbf{B} \rangle - \langle \mathbf{A} \rangle \mathbf{B} + \langle \mathbf{A} \rangle \langle \mathbf{B} \rangle - \mathbf{BA} + \mathbf{B}\langle \mathbf{A} \rangle + \langle \mathbf{B} \rangle - \langle \mathbf{A} \rangle \langle \mathbf{B} \rangle$$

Since expected values are just scalars, they commute with operators, and many cancelations take place, giving the result.
Last, by using Cauchy-Schwarz inequality $2|X||Y| \geq |\langle X|Y\rangle + \langle Y|X\rangle|$ and defining the states: $|X\rangle = \bar{\mathbf{A}}|\Psi\rangle$, $|Y\rangle = i\bar{\mathbf{B}}|\Psi\rangle$, one obtains the result wanted by following equations (5.11)$\rightarrow$(5.13) in Susskind.

g) For example, here we will show the commutation between σ_x, σ_y

$$[\sigma_x, \sigma_y] = \sigma_x \sigma_y - \sigma_y \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = 2i\sigma_z$$

Similarly we have

$$[\sigma_z, \sigma_x] = 2i\sigma_y$$
$$[\sigma_y, \sigma_z] = 2i\sigma_z$$

In general we can use the Levi-Cevita Tensor ϵ_{ijk} to arrive at a general equation

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k.$$

h) Since they do not commute they cannot be simultaneously observed as they do not share the same eigenbasis.
i) Since these operators are infinite-dimensional, they do not have a matrix representation and so we treat them as operators applied to the wavefunction $\psi(x)$

$$[\hat{x},\hat{p}]\psi(x) = \hat{x}\hat{p}\psi(x) - \hat{p}\hat{x}\psi(x) = \hat{x}(-i\hbar\frac{d}{dx}\psi(x)) - \hat{p}(x\psi(x)) = -i\hbar(x\frac{d}{dx}\psi(x)) + i\hbar\frac{d}{dx}(x\psi(x))$$

We perform the chain rule on the last term, which gives

$$[\hat{x},\hat{p}]\psi(x) = -i\hbar x\frac{d}{dx}\psi(x) + i\hbar\psi(x) + i\hbar x\frac{d}{dx}\psi(x) = i\hbar\psi(x).$$