Quantum Computing

Exercises 7: Quantum Fourier Transforms

The Quantum Fourier Transform acting on some state $|j\rangle$ is given by

$$|j\rangle \rightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i jk/N} |k\rangle.$$

$N = 2^n$ for n qubits. Or, in the tensor-product representation

$$|j\rangle \rightarrow \frac{1}{\sqrt{N}} \left(|0\rangle + e^{2\pi i \frac{j}{2^1}}|1\rangle\right) \otimes \cdots \otimes \left(|0\rangle + e^{2\pi i \frac{j}{2^n}}|1\rangle\right)$$

1. *How does the QFT act on $|0\rangle$ and $|1\rangle$? Find the QFT matrix for 1-qubit.*

2. *Show that the Quantum Fourier Transform acting on the n-qubit $|0\rangle^{\otimes n}$ state is equivalent to applying a Hadamard transform to each qubit, which we can write as*

$$H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} (-1)^{x \cdot k} |k\rangle.$$

3. *Directly proof that the general Quantum Fourier Transform is a unitary transformation.*

Hint: You may need to use the formula for a finite geometric series

$$\sum_{k=0}^{N-1} ar^k = a \left(\frac{1 - r^N}{1 - r}\right)$$

4. *Using both representations compute the output of applying the Quantum Fourier Transform on the state $|5\rangle_3$ ($n = 3$ qubits).*