Quantum Computing 2025 - Exercise Sheet 5

Introduction to Quantum Control

1. *(Time-independent systems)*
The time evolution of time-independent systems is modeled by the time-independent Schrödinger equation (TISE)

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle, \tag{1}$$

where $|\psi(t)\rangle$ is the quantum state at time t and H is the Hamiltonian of the system.

a) *The quantum states evolve through unitary matrices such that $|\psi(t)\rangle = U(t)|\psi(0)\rangle$. Why does $U(t)$ have to be unitary?*

b) *Rewrite the TISE in terms of $U(t)$.*

c) *Find the explicit formula for $U(t)$ that solves the TISE.*

d) *Show that if H is Hermitian then $U(t)$ is unitary.*

e) *If we let $H = \frac{\pi}{8}\sigma^z$, what is the state at time $t = 4\hbar$, $|\psi(4\hbar)\rangle = U(4\hbar)|\psi(0)\rangle$, where $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Use the formula $e^{-i\theta\sigma^z} = \cos(\theta)I - i\sin(\theta)\sigma^z$.*

2. *(Time-dependent systems)*
The time evolution of time-dependent systems is modeled by the time-dependent Schrödinger equation (TDSE)

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H(t)|\psi(t)\rangle. \tag{2}$$

If we consider controlled systems we can write the Hamiltonian as

$$H(t) = H_0 + u(t)H_c, \tag{3}$$

where H_0 is the free Hamiltonian, H_c is the control Hamiltonian, and $u(t)$ is a control function of any form.
The solution to the TDSE at some time T is also given by an exponential called the Magnus expansion, defined as

$$U(T) = \exp(\Omega^{(\infty)}), \qquad \Omega^{(n)} = \sum_{k=0}^{n}\Omega_k, \quad \textit{we approximately write (for some n)} \quad U(T) \simeq \exp(\Omega^{(n)}). \tag{4}$$

and we define the first terms of the sum as

$$\Omega_1(T) = \frac{-i}{\hbar}\int_0^T dt_1 H(t_1) \tag{5}$$

$$\Omega_2(T) = \frac{-1}{2\hbar}\int_0^T dt_1\int_0^{t_1} dt_2[H(t_1), H(t_2)]. \tag{6}$$

a) *If we let $H(t) = \sigma^z + t\sigma^x$ find $\Omega^{(2)}$ for $T = 1$ (You have to calculate both terms of the series and add them together).*

b) *What is $U(1)$ (approximate to the second order Magnus expansion)? Use the formula $e^{-i\theta(\mathbf{n}\cdot\sigma)} = \cos(\theta)I - i\sin(\theta)(\mathbf{n}\cdot\sigma)$ for* ***unit vector*** $\mathbf{n}$ *(make sure to normalize it and consider how this affects θ). Find the solution as $aI - i(b\sigma^x + c\sigma^y + d\sigma^z)$ for some constants a, b, c, d, and leave these in terms of cos and sin.*

3. *(Piecewise constant systems)*
If we divide the time into m segments $[t_1, t_2, \ldots, t_m = T]$ where the Hamiltonian is constant (time-independent). We can write the unitary evolution as

$$U(T) = e^{\frac{-i}{\hbar}H(t_m)\Delta t}\cdots e^{\frac{-i}{\hbar}H(t_2)\Delta t}e^{\frac{-i}{\hbar}H(t_1)\Delta t},$$

where $\Delta t = T/m$. If we let $T = 2\pi\hbar$ and $m = 4$ with equally long segments, find the unitary matrix that encodes the evolution of the system, $U(T)$, for Hamiltonian $H(t_1) = \sigma^z, H(t_2) = 2\sigma^y, H(t_3) = I, H(t_4) = \frac{1}{2}\sigma^x$.