Quantum Computing 2025 - Exercise Sheet 2

Deutsch–Jozsa's Algorithm

The Deutsch–Jozsa algorithm, formulated by David Deutsch and Richard Jozsa in 1992 and is the first example of Quantum algorithm providing an *exponential* speedup over a classical algorithm.

Problem Statement

You are given some unknown function (a black-box), $f(x)$. Here the input x is a bit-string of length n. In other words, x is a element of the $\{0, 1\}^n$ which has cardinality 2^n. The function will then output a single binary value i.e.

$$f(x) : \{0, 1\}^n \to \{0, 1\}$$

We are promised that this function will either be *constant* or *balanced*.

- A constant function will always output only 0 or only 1 for all x

- A balanced function will print 0 for exactly half of all inputs and 1 for the other half of the inputs.

The goal is to find out whether or not $f(x)$ is constant or balanced using as few queries as possible.

1. *(Classical Deutsch–Jozsa Algorithm)*

(a) *In the best possible case, how many inputs are needed to determine the type of function?*

(b) *In the worst possible case, how many inputs are needed? (with 100 % confidence rate)*

We will now go through the quantum algorithm, which only requires a single input! But first we need to introduce some other concepts.

2. *(Hadamard gate for n qubits) We have seen before that the Hadamard gate takes the computational basis $\{|0\rangle, |1\rangle\}$ to the equal superposition states $\{|+\rangle, |-\rangle\}$ (and vice versa). For multiple qubits starting in the computational basis, applying the Hadamard to all of them will create equal superposition between all the basis states, with a phase of ± 1 between the states. This can be time consuming to calculate the state, however there is a general formula one can use. For a bitstring x of length n, applying a Hadamard gate to each qubit (denoted as $H^{\otimes n}$) we find*

$$H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{k \in \{0,1\}^n} (-1)^{k \cdot x}|k\rangle$$

Using this formula find the result of applying $H^{\otimes n}$ to the following states

a) $|1\rangle$ b) $|01\rangle$ c) $|101\rangle$

3. *(Phase Kickback) Recall from the previous exercise sheet the CNOT gate. We saw that if the state of our control qubit is $|1\rangle$ we apply the X gate to our target qubit. In general we can talk about the Controlled-U gate which applies a phase $e^{i\phi}$ to the target qubit.*

(a) *Calculate $(Controlled\text{-}U)|1\rangle|\psi\rangle$*

In the above exercise, you should find that the state stays the same (since global phase changes does not affect quantum states). Now suppose we are in a superposition state

(b) *Calculate $(Controlled\text{-}U)|+\rangle|\psi\rangle$*

You should find here the result

$$\frac{1}{\sqrt{2}}(|0\rangle) + e^{i\phi}|1\rangle)|\psi\rangle$$

Amazingly, the change in a relative phase has been induced in the control qubit instead of the target. This is unique to quantum computing and is key to many algorithms.

(c) *Show $CNOT|+-\rangle$ also produces a phase kickback.*

4. *(The Duetsch- Jozsa Algorithm) To begin the algorithm we start with two qubit registers. The first is our query register which contains n qubits all in the 0 state (we write this as $|0\rangle^{\otimes n}$). The second contains a single ancilla qubit initialized in the state $|1\rangle$. Hence, our initial state is $|\psi_0\rangle = |0\rangle^{\otimes n}|1\rangle$*

(a) What is the state, $|\psi_1\rangle$, when we apply the Hadamard to every qubit?

Next we apply an oracle, O_f that implements our black-box function. The action of O_f on a $n+1$ qubit register is $|x\rangle^{\otimes n}|y\rangle \rightarrow |x\rangle^{\otimes n}|y \oplus f(x)\rangle$.

(b) Show that applying the oracle to $|\psi_1\rangle$ we induce a phase kickback and get the state

$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)}|x\rangle|-\rangle$$

Our ancilla qubit has now done it's job and we can discard it in the following.

(a) Apply Hadamards again to every qubit in the query register. What is the state $|\psi_3\rangle$? (Hint: write it as a double sum)

Finally, we now will measure the probability that all our qubits remained in the $|0\rangle^{\otimes n}$

(d) Show

$$P(|0\rangle^{\otimes n}) = \frac{1}{2^{2n}} \left| \sum_{x \in \{0,1\}^n} (-1)^{f(x)} \right|^2$$

Clearly we can see that if the $f(x)$ is constant then the we get the probability of being in $|0\rangle^{\otimes n}$ is 1. If $f(x)$ is balanced, we get in the summand that $(-1)^{f(x)} = -1$ for half and $(-1)^{f(x)} = 1$ for the other half which cancel each other out and hence the probability of remaining in the initial state is 0.

(e) Draw the circuit for the Deutsch-Josza algorithm.