Homework No. 07

Lecture introduced characteristic modes as solutions to specific eigenvalue problems. In this homework, the evaluation of characteristic modes of a cylindrical dipole is practiced.

Setup: Assume a dipole placed along the z-axis ($z \in (-L/2, L/2)$) which has a form of a thin-wall perfectly electrically conducting (PEC) tube. No cap is assumed at the end of the dipole. Assume that the dipole is described by an equivalent current density[1]

$$\boldsymbol{J}_{\mathrm{e}} = I(z) \frac{\delta(\rho - a)}{2\pi\rho} \mathbf{z}_0, \ z \in (-L/2, L/2). \tag{1}$$

Electrodynamics of this system is described by impedance matrix
$\mathbf{Z}_0 = \mathbf{R}_0 + \mathrm{j}\mathbf{X}_0$, which can be evaluated as

$$Z_{0,mn} = -\langle \boldsymbol{E}_{\mathrm{e}}(\boldsymbol{\psi}_m), \boldsymbol{\psi}_n \rangle, \tag{2}$$

where it is assumed that equivalent current density is expanded as

$$\boldsymbol{J}_{\mathrm{e}} = \sum_{n=1}^{N} I_n \boldsymbol{\psi}_n, \tag{3}$$

with

$$\boldsymbol{\psi}_n = \frac{\delta(\rho - a)}{2\pi\rho} \sin\left(n\pi\left(\frac{z}{L} - \frac{1}{2}\right)\right) \mathbf{z}_0, \ z \in (-L/2, L/2) \tag{4}$$

and where $\boldsymbol{E}_{\mathrm{e}}(\boldsymbol{J}_{\mathrm{e}})$ is electric field generated by current density $\boldsymbol{J}_{\mathrm{e}}$. The current expansion (3) is equivalent to

$$I(z) = \sum_{n=1}^{N} I_n f_n(z) \tag{5}$$

with

$$f_n(z) = \sin\left(n\pi\left(\frac{z}{L} - \frac{1}{2}\right)\right), \ z \in (-L/2, L/2). \tag{6}$$

Based on the previous homework, there is

$$\mathbf{z}_0 \cdot \boldsymbol{E}_{\mathrm{e}}(\boldsymbol{\psi}_m, \rho > a, z) = -\frac{kZ}{8\pi} \int\limits_{-\infty}^{\infty} \left(1 - \frac{k_z^2}{k^2}\right) \tilde{f}_m(k_z) \, \mathrm{H}_0^{(2)}(k_\rho \rho) \, \mathrm{J}_0(k_\rho a) \, \mathrm{e}^{\mathrm{j}k_z z} \mathrm{d}k_z, \tag{7}$$

where Z is the free-space impedance, $\tilde{f}_m(k_z) = \mathcal{F}_z\{f_m(z)\}$ is the Fourier's transform of the current basis function, $\mathrm{H}_n^{(2)}$ is Hankel's function of second kind and order n and $k_\rho^2 = k^2 - k_z^2$ with $\mathrm{Im}\{k_\rho\} < 0$. This leads to

$$Z_{0,mn} = \frac{kZ}{8\pi} \int\limits_{-\infty}^{\infty} \tilde{f}_m(k_z) \tilde{f}_n^*(k_z) \left(1 - \frac{k_z^2}{k^2}\right) \mathrm{H}_0^{(2)}(k_\rho a) \, \mathrm{J}_0(k_\rho a) \, \mathrm{d}k_z. \tag{8}$$

[1] It is assumed that potential excitation does not break the angular symmetry.

Task No. 1: Knowing the impedance matrix $\mathbf{Z}_0$, evaluate characteristic modes

$$\mathbf{X}_0 \mathbf{I} = \lambda \mathbf{R}_0 \mathbf{I}. \tag{9}$$

Plot current profiles of the first five characteristic modes at electrical size $kL \approx 0.9\pi$ and ratio $L/a = 50$. Comment on their composition via basis functions. Numerically check the orthogonality relations

$$\begin{aligned} \mathbf{I}_m^{\mathrm{H}} \mathbf{R}_0 \mathbf{I}_n &= \mathbf{I}_n^{\mathrm{H}} \mathbf{R}_0 \mathbf{I}_n \delta_{mn}, \\ \mathbf{I}_m^{\mathrm{H}} \mathbf{X}_0 \mathbf{I}_n &= \mathbf{I}_n^{\mathrm{H}} \mathbf{X}_0 \mathbf{I}_n \delta_{mn}. \end{aligned} \tag{10}$$

Task No. 2: Try to evaluate frequency sweep of the first five (smallest in magnitude) characteristic numbers λ for electrical size $kL \in (0.1, 2.0)\,\pi$ and the same ratio L/a as above. Notice that when characteristic number is positive / negative, the mode predominantly carries magnetic / electric energy (inductive / capacitive mode). When characteristic number passes through zero, the mode is at resonance. At that frequency, the excited current distribution is typically dominated by this characteristic mode.