Homework No. 05

Lecture discussed the use of volume and surface equivalent on the analysis of cylindrical monopole antenna over a ground plane that is fed by a connected coaxial line. This homework recapitulates the description and addresses the solution of input impedance via field integral equation.

Setup: In order to simplify the analysis, let us assume that only the dominant TEM mode exists in the coaxial aperture. That is, the electric field in the coaxial aperture reads

$$\boldsymbol{E} = \boldsymbol{\rho}_0 \frac{U}{\rho \ln \dfrac{b}{a}}, \tag{1}$$

where a, b is the inner and outer radius of the coaxial line, U is the voltage at the aperture and where coaxial line runs along z-axis. The monopole is assumed to be a continuation of the inner coaxial conductor ($z \in (0, L/2)$) which has a form of a thin-wall perfectly electrically conducting (PEC) tube. No cap is assumed at the end of the monopole. The ground plane is assumed to be perfectly conducting and positioned at $z = 0$ plane.

In order to obtain the integral equation for the current on the monopole, a surface equivalent is applied on the half space $z < 0$ with a PEC filling while volume equivalence (or surface equivalence with vacuum filling) is applied on the monopole. Applying further image theory over the PEC surface at $z = 0$ we are finally left with an electric current

$$\boldsymbol{J}_{\mathrm{e}} = I\left(z\right) \frac{\delta\left(\rho - a\right)}{2\pi\rho} \mathbf{z}_0,\ z \in \left(-L/2, L/2\right), \tag{2}$$

which is at xy plane encircled by a magnetic current

$$\boldsymbol{J}_{\mathrm{m}} = -\frac{2U}{\rho \ln \dfrac{b}{a}} \delta\left(z\right) \boldsymbol{\varphi}_{\mathbf{0}},\ \rho \in \left(a, b\right). \tag{3}$$

Notice that the current $I\left(z\right)$ on the monopole (due to reflection it is actually a dipole) must be an even function due to the applied reflection through the PEC plane.

The current $I\left(z\right)$ is unknown and is to be obtained from boundary condition on electric field over the surface of the monopole (dipole) tube. Particularly, there must be

$$\boldsymbol{\rho}_0 \times \Big[\boldsymbol{E}_{\mathrm{e}}\left(\rho = a, z\right) + \boldsymbol{E}_{\mathrm{m}}\left(\rho = a, z\right)\Big] = 0,\ z \in \left(-L/2, L/2\right), \tag{4}$$

where it is important to realize that current density $\boldsymbol{J}_{\mathrm{e}}$ and $\boldsymbol{J}_{\mathrm{m}}$ radiate in free space and, therefore, the electric field $\boldsymbol{E}_{\mathrm{e}}, \boldsymbol{E}_{\mathrm{m}}$ produced by them is relatively easy to find.

Task No. 1: Study [1, 4.8] and show that

$$\mathbf{z}_0 \cdot \boldsymbol{E}_{\mathrm{m}} = \frac{\mathrm{j}U}{\ln \frac{b}{a}} \int\limits_0^{\infty} \frac{\mathrm{e}^{-\mathrm{j}k_z|z|}}{k_z} \Big[\mathrm{J}_0\left(k_\rho b\right) - \mathrm{J}_0\left(k_\rho a\right) \Big] \mathrm{J}_0\left(k_\rho \rho\right) k_\rho \mathrm{d}k_\rho, \tag{5}$$

where $k_z^2 = k^2 - k_\rho^2$, $\mathrm{Im}\left\{k_z\right\} < 0$, k denotes free-space wavenumber and J_n denotes Bessel's function of order n.

Task No. 2: Use Fourier's transform along z on Helmholtz's equation for magnetic vector potential generated by current $\boldsymbol{J}_{\mathrm{e}}$ together with [1, 4.7] and show that

$$\mathbf{z}_0 \cdot \boldsymbol{E}_{\mathrm{e}}\left(\rho > a, z\right) = -\frac{kZ}{8\pi} \int\limits_{-\infty}^{\infty} \left(1 - \frac{k_z^2}{k^2}\right) \tilde{I}\left(k_z\right) \mathrm{H}_0^{(2)}\left(k_\rho \rho\right) \mathrm{J}_0\left(k_\rho a\right) \mathrm{e}^{\mathrm{j}k_z z} \mathrm{d}k_z, \tag{6}$$

where Z is the free-space impedance, $\tilde{I}\left(k_z\right) = \mathcal{F}_z\left\{I\left(z\right)\right\}$ is the Fourier's transform of the current on the monopole (dipole), $\mathrm{H}_n^{(2)}$ is Hankel's function of second kind and order n and $k_\rho^2 = k^2 - k_z^2$ with $\mathrm{Im}\left\{k_\rho\right\} < 0$.

Task No. 3: Use (4) to formulate integral equation for unknown current $I\left(z\right)$ and then apply method of moments to solve this equation using expansion

$$I\left(z\right) = \sum_{n=1,3,5,\ldots} I_n \sin\left(n\pi\left(\frac{z}{L} - \frac{1}{2}\right)\right). \tag{7}$$

Evaluate input impedance of the monopole as seen from the coaxial line as

$$Z_{\mathrm{in}} = \frac{U}{I\left(0\right)}. \tag{8}$$

For numerical evaluation use $kL \approx 0.9\pi$, $L/a = 50$ and $b/a = 2.3$. You should observe that $Z_{\mathrm{in}} \approx 37\,\Omega$. Plot the current $I\left(z\right)$ along the dipole under these conditions.

References

[1] D. G. Dudley, *Mathematical Foundations for Electromagnetic Theory*. IEEE Press, 1994.