Homework No. 04

The lecture discussed the use of Lorentz's reciprocity to evaluate the current density on a receiving antenna from the knowledge of the current density on the same antenna in transmitting mode. Use the lecture notes to solve this homework.

Task No. 1: A loop of radius a is made of a perfectly conducting wire of negligible thickness and is excited by an incident plane wave

$$\boldsymbol{E}_{\mathrm{i}} = \mathbf{y}_0 E_0 \mathrm{e}^{-\mathrm{j}kx}, \tag{1}$$

where k is the free-space wavenumber. The loop is loaded by an impedance

$$Z_{\mathrm{load}} = \frac{1}{\mathrm{j}\omega C_{\mathrm{load}}} \tag{2}$$

according to Fig. 1.

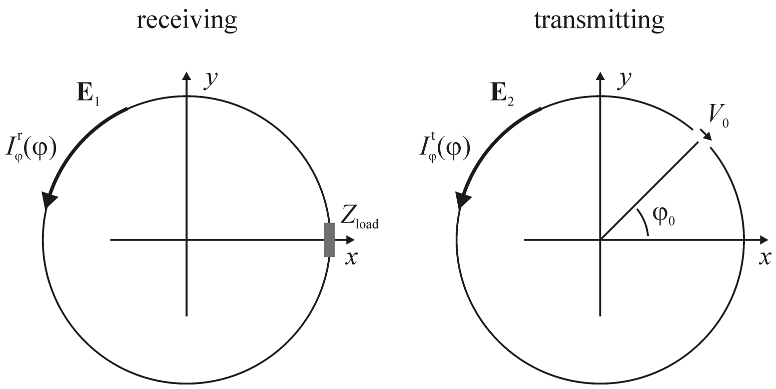

Figure 1: A sketch of a loop made of thin perfectly conducting wire. The left panel shows a loop in receiving mode, while the right panel shows a loop in transmitting mode. The receiving loop is loaded by impedance Z_{load}. The transmitting loop is fed by a delta gap source with voltage V_0.

Evaluate electric and magnetic dipole moment induced on the loop

$$\begin{aligned} \boldsymbol{p} &= \frac{1}{\mathrm{j}\omega} \int\limits_V \boldsymbol{J}^{\mathrm{r}} \mathrm{d}V \\ \boldsymbol{m} &= \frac{1}{2} \int\limits_V \boldsymbol{r} \times \boldsymbol{J}^{\mathrm{r}} \mathrm{d}V \end{aligned} \tag{3}$$

assuming that $ka \ll 1$. Under this assumption, the current existing on the transmitting loop can be approximated as

$$I^{\mathrm{t}}_{\varphi}(\varphi) = V_0 \left[\frac{1}{\mathrm{j}\omega L_0} + \mathrm{j}\omega C_0 \cos(\varphi - \varphi_0) \right], \tag{4}$$

where C_0, L_0 are self-capacitance and self-inductance of the loop, respectively.

Hint: To simplify the evaluation, expand the incident electric field into Taylor's series at origin and keep only the first two non-zero terms.

Result:

$$
\begin{aligned}
\boldsymbol{p} &= \mathbf{y}_0 \frac{a^2\pi^2}{\frac{\omega_0^2}{\omega^2}-1}\left(-\frac{\omega_0^2 C_0 a}{\mathrm{j}\omega} B_0 + \frac{C_{\text{load}} C_0}{C_{\text{load}}+C_0}\left(\frac{\omega_{\text{zero}}^2}{\omega^2}-1\right) E_0\right) \\
\boldsymbol{m} &= \mathbf{z}_0 \frac{a^2\pi^2}{\frac{\omega_0^2}{\omega^2}-1}\left(\frac{a^2 B_0}{L_0} + \frac{\omega_0^2 C_0 a}{\mathrm{j}\omega} E_0\right),
\end{aligned}
\tag{5}
$$

where

$$
\begin{aligned}
\omega_0^2 &= \frac{1}{L_0\left(C_{\text{load}}+C_0\right)} \\
\omega_{\text{zero}}^2 &= \frac{1}{L_0 C_{\text{load}}}.
\end{aligned}
\tag{6}
$$

Notice the antisymmetry of electric-magnetic and magnetic-electric couplings in the dipole moments resulting from reciprocity.