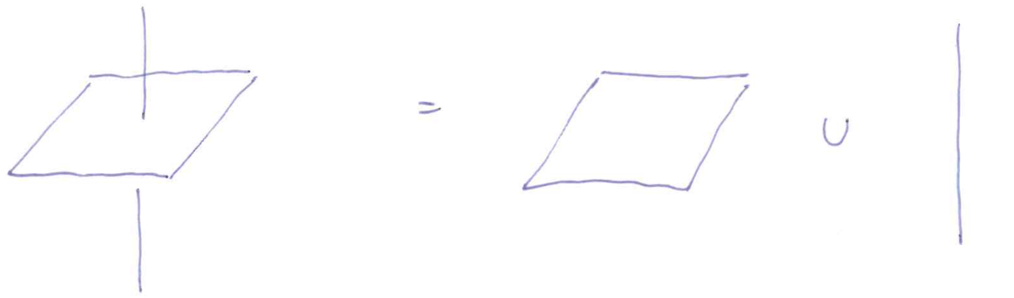
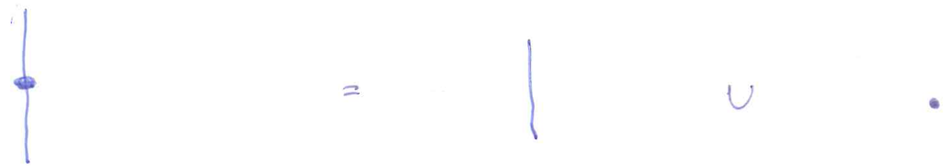
Primary & prime decomposition

Complex (complicated) varieties can be decomposed



$$V(\langle xz, yz\rangle) = V(\langle z\rangle) \cup V(\langle x, y\rangle)$$



$$\underbrace{\langle x^2, xy\rangle} \qquad V(\langle x\rangle) \qquad V(\langle x, y\rangle)$$

Ideal $\langle x^2, xy\rangle$ encodes more than it's variety $V(\langle x^2, xy\rangle) = V(\sqrt{\langle x^2, xy\rangle}) = V(\langle x\rangle)$

Integers

Fundamental theorem of arithmetics:

Every positive integer $n > 1$ can be represented in exactly one way as a product of prime powers

$$n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$$

where $p_1 < p_2 < \dots p_k$ are primes and n_i are positive integers.

Integers $\mathbb{Z}$ form a ring and it's ideals are of the form $\langle n \rangle$, $n \in \mathbb{Z}$.

<u>Prime ideals</u>

$\langle p \rangle$, p prime are prime ideals.

We see that $\langle p \rangle = \{ c \cdot p \mid c \in \mathbb{Z} \}$ and thus

$a \cdot b \in \langle p \rangle \Leftrightarrow \exists c \in \mathbb{Z} : a \cdot b = p \cdot c$

implies $p \mid a$ or $p \mid b$ or, equivalenty,

$a \in \langle p \rangle$ or $b \in \langle p \rangle$ (proof: FTA: write all uniquely using primes)

This serves as a general definition:

<u>Definition (Prime ideal)</u>: A proper ideal I in a commutative ring R is <u>prime</u> if there holds true

$a, b \in R \wedge a \cdot b \in I \Rightarrow a \in I \vee b \in I$

<u>Examples</u>:

$\langle x \rangle$	$\langle x^2 \rangle$	$x \notin \langle x^2 \rangle$
YES: prime	NOT prime	$x \cdot x \in \langle x^2 \rangle$

<u>Back to $\mathbb{Z}$</u>:

$$n = p_1^{n_1} \cdot p_2^{n_2} \cdots p_k^{n_k} \quad \begin{cases} n_1 = n_2 = \cdots = n_k = 1 & \text{Prime} \\ \exists i : n_i > 1 & \text{NOT Prime} \end{cases}$$

Why are prime ideals useful?

<u>Theorem</u>: An affine variety $V \subseteq \mathbb{C}^n$ is irreducible if and only if $I(V)$ is a prime ideal.

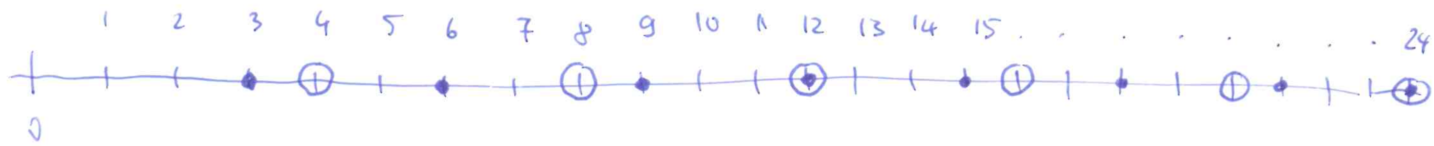
Theorem: Any variety in $\mathbb{C}^n$ can be uniquely represented as a finite union of irreducible varieties.

But even in integers, we get

$$\langle n \rangle = \underbrace{\langle p_1^{n_1} \rangle}_{\text{not prime for } n_1 > 1} \cap \langle p_2^{n_2} \rangle \cap \dots \cap \langle p_k^{n_k} \rangle$$

Example: $\langle 12 \rangle = \langle 3 \rangle \cap \langle 2^2 \rangle$

$\{12, 24, \dots\}$ $\{3, 6, 9, 12, \dots\}$ $\{4, 8, 12, \dots\}$



$\langle p_1^{n_1} \rangle$ are <u>primary</u>

<u>Definition (Primary ideal)</u>: A proper ideal I in a commutative ring R is <u>primary</u> if there holds true

$$a, b \in R \wedge a \cdot b \in I \Rightarrow a \in I \vee \exists n \in \mathbb{Z} : b^n \in I.$$

<u>Examples</u>

$\langle x \rangle$ prime $\Downarrow$ primary

$\langle x^2 \rangle$ primary $x \cdot x = x^2 \in \langle x^2 \rangle$

$\langle x^2, xy \rangle = I$ NOT primary

× $x \cdot y \in I$, $x \notin I \wedge \nexists n : y^n \in I$

✓ $y \cdot x \in I$, $y \notin I \wedge x^2 \in I$

<u>Theorem</u>: Let I be an ideal in $\mathbb{C}[x_1, \dots, x_n]$. Then there exist primary ideals $I_1, \dots, I_k$ such that $I = I_1 \cap I_2 \cap \dots \cap I_k$ and

minimal primary decomposition:

1) $\sqrt{I_i} \neq \sqrt{I_j}$ for $i, j = 1, \dots, k$, $i \neq j$
(pairwise distinct radicals)

2) $\bigcap_{j \neq i} I_j \not\subset I_i$ for all $1 \leq i \leq k$
(irredundant intersection)

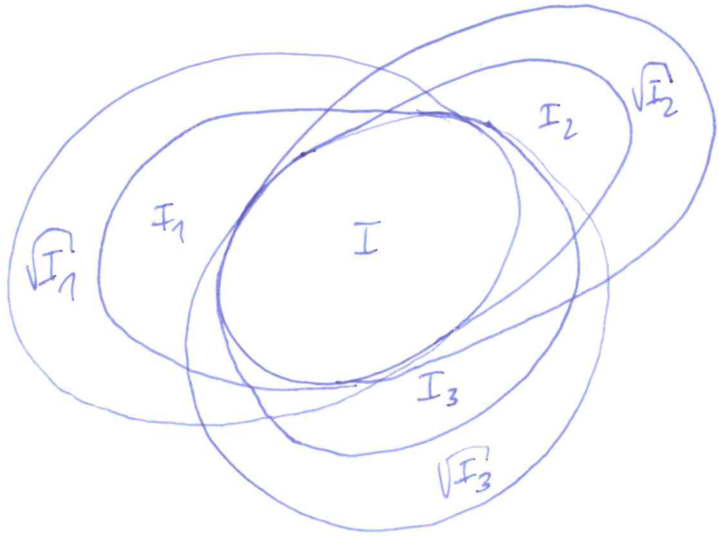
Primary decomposition is not unique

Example: $\langle x^2, xy \rangle = \langle x \rangle \cap \underbrace{\langle x, y \rangle^2}_{} = \langle x \rangle \cap \langle x^2, y \rangle$, where $\langle x, y \rangle^2 = \langle x^2, xy, y^2 \rangle$

$$\sqrt{\langle x \rangle} \qquad \sqrt{\langle x, y \rangle^2} \qquad \sqrt{\langle x \rangle} \qquad \sqrt{\langle x^2, y \rangle}$$

$$\{ \langle x \rangle \quad \langle x, y \rangle \} = \{ \langle x \rangle \, , \, \langle x, y \rangle \}$$

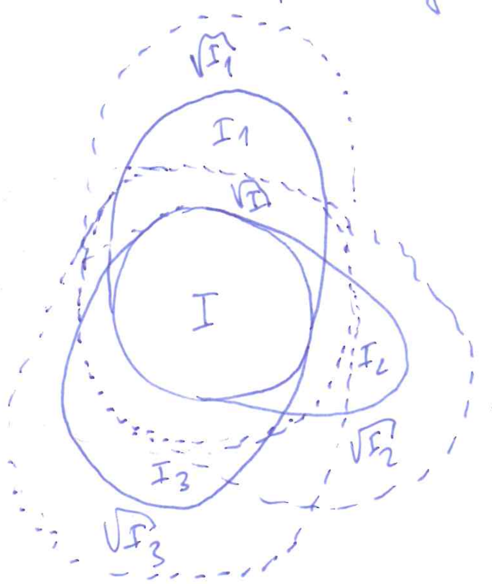
3) The set $\{ \sqrt{I_i} \}_{i=1}^{k}$ of prime ideals is unique.



Picture of $I = \underbrace{I_1 \cap I_2 \cap I_3}_{\text{primary}}$ & $\sqrt{I} = \underbrace{\sqrt{I_1} \cap \sqrt{I_2} \cap \sqrt{I_3}}_{\text{prime}}$

$\downarrow$

irreducible decomposition of $V(I)$



Embedded & minimal primes

$$\langle x^2, xy\rangle = \langle x\rangle \cap \langle x^2, y\rangle$$

$$\begin{array}{ccc} & \downarrow & & \downarrow & \\ & \sqrt{\langle x\rangle} & & \sqrt{\langle x^2, y\rangle} & \\ & \| & & \| & \\ \text{minimal}: & \langle x\rangle & \subseteq & \langle x, y\rangle & : \text{Embedded} \end{array}$$