

## PKR Lab-07 Solution

**Task 1.** Consider the rotation matrix with rotation axis generated by vector  $\mathbf{r} = [1 \ 1 \ 1]^\top$  that maps vector  $\mathbf{x} = [0 \ 0 \ 1]^\top$  to  $\mathbf{y} = \mathbf{R}\mathbf{x} = [1 \ 0 \ 0]^\top$ .

(a) Find its rotation angle  $-\pi < \theta \leq \pi$ ,

(b) Find its rotation matrix  $\mathbf{R}$ .

**Solution:** We use the angle-axis parametrization of the rotation [1, Equation (7.22)]:

$$\mathbf{R} = \cos \theta \mathbf{I} + (1 - \cos \theta) \mathbf{v} \mathbf{v}^\top + \sin \theta [\mathbf{v}]_\times \quad (1)$$

where  $\mathbf{v} = \frac{1}{\sqrt{3}} [1 \ 1 \ 1]$  is the normalized axis of rotation (the sign of  $\mathbf{v}$  is fixed by our choice). By the task,  $\mathbf{R} [0 \ 0 \ 1]^\top = [1 \ 0 \ 0]^\top$ . Hence, multiplying both sides of Equation (1) by  $[0 \ 0 \ 1]^\top$  we get

$$\begin{aligned} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &= \mathbf{R} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = (\cos \theta \mathbf{I} + (1 - \cos \theta) \mathbf{v} \mathbf{v}^\top + \sin \theta [\mathbf{v}]_\times) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &= \cos \theta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{1 - \cos \theta}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{\sin \theta}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \end{aligned} \quad (2)$$

From the last equation of Equation (2)

$$0 = \cos \theta + \frac{1 - \cos \theta}{3}$$

we can express

$$\cos \theta = -\frac{1}{2}.$$

Substituting it to the second equation in (2) we get

$$0 = \frac{1}{2} - \frac{\sin \theta}{\sqrt{3}}$$

from which we get

$$\sin \theta = \frac{\sqrt{3}}{2}$$

The rotation angle then equals

$$\theta = \text{atan2}(\sin \theta, \cos \theta) = \text{atan2}\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \frac{2\pi}{3}.$$

We get the rotation matrix by substituting  $\mathbf{v}$  and  $\theta$  to Equation (1):

$$\mathbf{R} = -\frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Another pair of rotation angle and axis which generates the same  $\mathbf{R}$  is  $(-\theta, -\mathbf{v})$ . □

**Task 2.** Consider unit quaternion

$$\mathbf{q} = \frac{1}{3} [0 \ -1 \ -2 \ -2]$$

(a) For the rotation given by  $\mathbf{q}$ , find all pairs of  $(\theta, \mathbf{v})$  corresponding to its rotation angle  $-\pi < \theta \leq \pi$  and its rotation axis generated by unit vector  $\mathbf{v}$ ,

(b) Find the rotation matrix corresponding to  $\mathbf{q}$ .

**Solution:**

- (a) The quaternion is defined by

$$\mathbf{q} = \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \mathbf{v} \end{bmatrix}$$

where  $(\theta, \mathbf{v})$  is the pair of a rotation angle and a normalized axis of rotation. We have

$$\cos \frac{\theta}{2} = 0 \Rightarrow \sin \frac{\theta}{2} = \pm 1$$

We, e.g., take the pair  $(\cos \frac{\theta}{2}, \sin \frac{\theta}{2}) = (0, 1)$  which gives

$$\frac{\theta}{2} = \frac{\pi}{2} + 2\pi k, k \in \mathbb{Z} \iff \theta = \pi + 4\pi k, k \in \mathbb{Z}$$

By the task we want  $-\pi < \theta \leq \pi$ , so  $\theta = \pi$ . We compute the normalized axis of rotation  $\mathbf{v}$  by dividing the last 3 coordinates of  $\mathbf{q}$  by  $\sin \frac{\theta}{2}$ :

$$\mathbf{v} = \frac{1}{3} \begin{bmatrix} -1 & -2 & -2 \end{bmatrix}^\top.$$

If  $(\theta, \mathbf{v})$  defines  $\mathbf{q}$  for  $-\pi < \theta \leq \pi$ , then all pairs  $(\theta, \mathbf{v})$  with  $-\pi < \theta \leq \pi$  that define the rotation given by  $\mathbf{q}$  are determined by  $\{(\theta, \mathbf{v}), (-\theta, -\mathbf{v})\}$ . Since for  $\theta = \pi$  the value  $-\theta = -\pi$  jumps out of the interval  $(-\pi, \pi]$ , then we simply add  $2\pi$  to it, since it doesn't change the rotation matrix (according to the Rodriguez formula [1, Equation 7.22]). Hence, the answer is

$$\left\{ \left( \pi, -\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}^\top \right), \left( \pi, \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}^\top \right) \right\}.$$

- (b) The rotation matrix is given by the Rodriguez formula [1, Equation 7.22]:

$$\begin{aligned} \mathbf{R} &= \cos \theta \mathbf{I} + (1 - \cos \theta) \mathbf{v} \mathbf{v}^\top + \sin \theta [\mathbf{v}]_\times \\ \mathbf{R} &= -\mathbf{I} + 2 \cdot \frac{1}{9} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} -1 & & \\ & -1 & \\ & & -1 \end{bmatrix} + \frac{2}{9} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} -7 & 4 & 4 \\ 4 & -1 & 8 \\ 4 & 8 & -1 \end{bmatrix} \end{aligned}$$

Another way to obtain  $\mathbf{R}$  is to use the formula in terms of quaternions [1, Equation 7.67]:

$$\mathbf{R} = \begin{bmatrix} q_1^2 + q_2^2 - q_3^2 - q_4^2 & 2(q_2q_3 - q_1q_4) & 2(q_2q_4 + q_1q_3) \\ 2(q_2q_3 + q_1q_4) & q_1^2 - q_2^2 + q_3^2 - q_4^2 & 2(q_3q_4 - q_1q_1) \\ 2(q_2q_4 - q_1q_3) & 2(q_3q_4 + q_1q_2) & q_1^2 - q_2^2 - q_3^2 + q_4^2 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} -7 & 4 & 4 \\ 4 & -1 & 8 \\ 4 & 8 & -1 \end{bmatrix}$$

□

**Task 3.** Consider rotation matrix

$$\mathbf{R} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

- (a) Find its (unit) rotation axis and angle  $-\pi < \theta \leq \pi$ .  
(b) Find all unit quaternions corresponding to  $\mathbf{R}$ .

**Solution:**

- (a) The rotation axis is given by the eigenvector corresponding to the eigenvalue  $\lambda = 1$ :

$$\mathbf{R}\mathbf{v} = \mathbf{v} \iff (\mathbf{R} - \mathbf{I})\mathbf{v} = \mathbf{0}$$

We solve the linear homogeneous system of equations:

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We obtain 1 zero row in the row echelon form of  $\mathbf{R} - \mathbf{I}$  indicating that  $\dim \ker(\mathbf{R} - \mathbf{I}) = 1$ . Let's denote  $\mathbf{v} = [v_1 \ v_2 \ v_3]$ . Then we let  $v_3$  to be any real number  $t$ . From the second equation  $-v_2 + v_3 = 0$  we obtain  $v_2 = v_3 = t$ . From the first  $-v_1 + v_2 = 0$  we obtain  $v_1 = v_2 = t$ . Thus, all the solutions to this linear system may be described by

$$\left\{ \begin{bmatrix} t \\ t \\ t \end{bmatrix} \mid t \in \mathbb{R} \right\}$$

Out of this set we take one of unit norm, e.g.,

$$\mathbf{v} = -\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The rotation angle can be determined from the Rodriguez formula:

$$\mathbf{R} = \cos \theta \mathbf{I} + (1 - \cos \theta) \mathbf{v} \mathbf{v}^\top + \sin \theta [\mathbf{v}]_\times$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \cos \theta \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + \frac{1}{3}(1 - \cos \theta) \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \frac{1}{\sqrt{3}} \sin \theta \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Out of these 9 equations we pick 2 given by the elements (1, 1) and (1, 2):

$$(1, 1): \quad 0 = \cos \theta + \frac{1}{3}(1 - \cos \theta) \iff \cos \theta = -\frac{1}{2}$$

$$(1, 2): \quad 1 = \frac{1}{3}(1 - \cos \theta) + \frac{1}{\sqrt{3}} \sin \theta \iff \sin \theta = \frac{\sqrt{3}}{2}$$

from which we deduce that  $\theta = \frac{2\pi}{3}$ . Another way to compute the rotation axis and angle is to apply the formula [1, Equations 7.40, 7.41] for non-symmetric rotations:

$$\theta = \arccos \left( \frac{1}{2} \text{trace } \mathbf{R} - 1 \right) = \frac{2\pi}{3}, \quad \mathbf{v} = \frac{1}{2 \sin \theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} = -\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Notice that for a symmetric rotation (i.e., a rotation by  $\pi$ )  $\mathbf{v}$  is undefined, so this formula is not applicable. As was noted before, all the angle-axis (for  $-\pi < \theta \leq \pi$ ) solutions to  $\mathbf{R}$  are given by  $\{(\theta, \mathbf{v}), (-\theta, -\mathbf{v})\}$ .

(b) One way is to apply the formula

$$\mathbf{q}_{1,2} = \pm \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \mathbf{v} \end{bmatrix} = \pm \begin{bmatrix} \cos \frac{\pi}{3} \\ \sin \frac{\pi}{3} \mathbf{v} \end{bmatrix} = \pm \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix} = \pm \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

The other way is to use [1, Equation 7.74]:

$$\mathbf{q}_{1,2} = \pm \frac{1}{2\sqrt{\text{trace } \mathbf{R} + 1}} \begin{bmatrix} \text{trace } \mathbf{R} + 1 \\ r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} = \pm \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

Again, the last formula works only for non-symmetric rotations.

□

## References

- [1] Tomas Pajdla, *Elements of geometry for robotics*, [https://cw.fel.cvut.cz/b221/\\_media/courses/pkr/pro-lecture-2021.pdf](https://cw.fel.cvut.cz/b221/_media/courses/pkr/pro-lecture-2021.pdf).