PKR Lab-06 Solution

Task 1. *Consider motion given by a mapping of a general point X to point Y by*

$$\vec{y}_\beta = \mathbf{R}\,\vec{x}_\beta + \vec{o}\,'_\beta\,, \tag{1}$$

where $\vec{x}_\beta$, resp. $\vec{y}_\beta$, are coordinate vectors representing point X, resp. point Y, in a coordinate system with an orthonormal basis β. Matrix $\mathbf{R}$ and vector $\vec{o}\,' = \overrightarrow{OO'}$ are given as follows

$$\mathbf{R} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad \vec{o}'_\beta = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

(a) Write down the matricidal equation determining the coordinates of points on the axis of motion.

(b) Find all the points on the axis of motion.

(c) Compare the axis of motion to the line obtained by translating the rotation axis by $\vec{o}'_\beta$.

Solution: (a) The axis of motion is the line in $\mathbb{R}^3$ that is left invariant after applying Equation (1). The matrix equation that determines the points on the axis of motion is [1, Equation (9.2)]:

$$(\mathbf{R}-\mathbf{I})^2\vec{x}_\beta = -(\mathbf{R}-\mathbf{I})\vec{o}\,'_\beta$$

(b) Substituting $\mathbf{R}$ and $\vec{o}'_\beta$ to it we obtain

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}^2 \vec{x}_\beta = -\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{bmatrix}\vec{x}_\beta = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \tag{2}$$

We can solve this system of linear equations by Gaussian elimination:

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \\ -2 & 1 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -2 & 1 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -2 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Denote $\vec{x}_\beta = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top$. From the second row $x_2 - x_3 = 0$ we conclude that $x_2 = x_3$ and we let x_3 to be any real number t. From the first row $x_1 - 2x_2 + x_3 = 1$ we conclude that $x_1 = 2x_2 - x_3 + 1 = t + 1$. Thus, the solutions to (2) are

$$L = \left\{ \begin{bmatrix} t+1 \\ t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

We may verify that the line L is invariant under the motion given by (1). For this we pick a general point from L and substitute it into (1):

$$\mathbf{R}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) + \vec{o}\,'_\beta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We see that L is invariant (in this case even point-wise, since $\vec{o}\,'_\beta$ is perpendicular to the rotation axis of $\mathbf{R}$).

(c) The rotation axis translated by $\vec{o}'_\beta$ has the form

$$L' = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\},$$

which is different from L, because

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \in L', \qquad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \notin L.$$

□

References

[1] Tomas Pajdla, *Elements of geometry for robotics*, `https://cw.fel.cvut.cz/b221/_media/courses/pkr/pro-lecture-2021.pdf`.