Analytical Modeling of Parallel Systems

Ananth Grama, Anshul Gupta, George Karypis, and Vipin Kumar

To accompany the text ``Introduction to Parallel Computing'', Addison Wesley, 2003.

Topic Overview

- Sources of Overhead in Parallel Programs
- Performance Metrics for Parallel Systems
- Effect of Granularity on Performance
- Scalability of Parallel Systems
- Asymptotic Analysis of Parallel Programs

Analytical Modeling - Basics

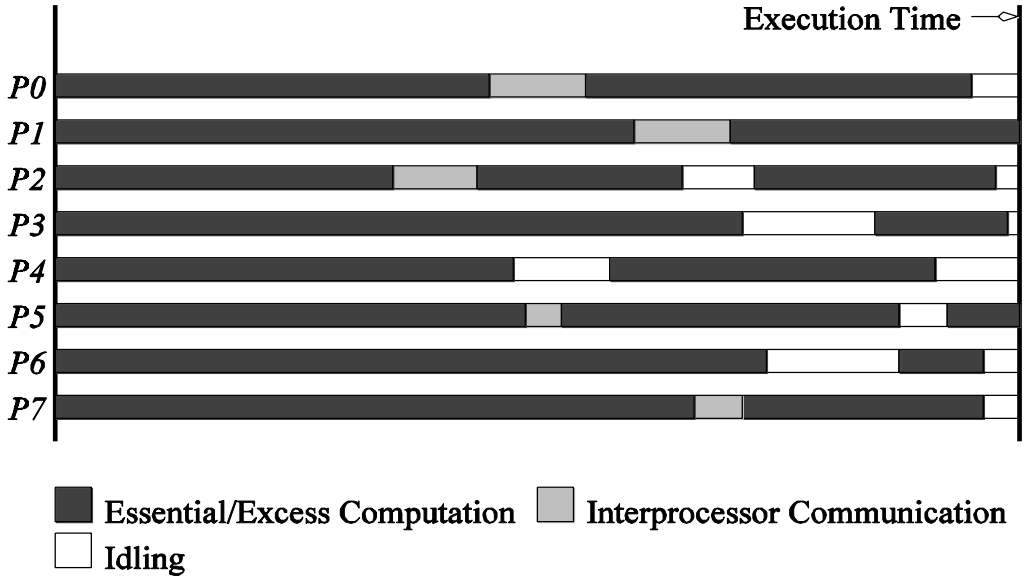
- A **sequential algorithm** is evaluated by its runtime (in general, **asymptotic runtime** as a function of input size).
- The asymptotic runtime of a sequential program is **identical on any serial platform**.
- The **parallel runtime** of a program depends on the **input size**, the **number of processors**, and the **communication parameters** of the machine.
- An algorithm must therefore be analyzed in the context of the **underlying platform**.
- A parallel system is a combination of a parallel algorithm and an underlying platform.

Analytical Modeling - Basics

- A number of performance measures are intuitive.
- **Wall clock time -** the time from the start of the first processor to the stopping time of the last processor in a parallel ensemble. **But how does this scale when the number of processors is changed** of the program is ported to another machine altogether?
- **How much faster is the parallel version?** This begs the obvious followup question - whats the baseline serial version with which we compare? Can we use a suboptimal serial program to make our parallel program look
- **Raw FLOPS (FLoating-point Operations Per Second) -** How good is FLOPS measure when it don't solve a problem?

Sources of Overhead in Parallel Programs

- If I use two processors, shouldnt my program run twice as fast?
- No - a number of overheads, including **wasted computation**, **communication**, **idling**, and **contention** cause degradation in performance.



The execution profile of a hypothetical parallel program executing on eight processing elements. Profile indicates times spent performing computation (both essential and excess), communication, and idling.

Sources of Overheads in Parallel Programs

- **Interprocess interactions**: Processors working on any non-trivial parallel problem will **need to talk to each other**.
- **Idling**: Processes may idle because of **load imbalance**, **synchronization**, or **serial components**.
- **Excess Computation**: This is computation **not performed by the serial version**. This might be because the **serial algorithm is difficult to parallelize**, or that some computations are repeated across processors **to minimize communication**.

Performance Metrics for Parallel Systems: Execution Time

- **Serial runtime** of a program is the **time elapsed between the beginning and the end** of its execution on a sequential computer.
- **The parallel runtime** is the time that elapses from the moment the **first processor starts** to the moment the **last processor finishes** execution.
- We denote the serial runtime by T_S and the parallel runtime by T_P.

Performance Metrics for Parallel Systems: Total Parallel Overhead

- Let T_{all} be the total time collectively spent by all the processing elements.
- T_S is the serial time.
- Observe that $T_{all} - T_S$ is then the total time spend by all processors combined in **non-useful work**. This is called the ***total overhead***.
- The total time collectively spent by all the processing elements
  $T_{all} = p\,T_P$ (p is the number of processors).
- The overhead function (T_o) is therefore given by

$$T_o = p\,T_P - T_S \qquad (1)$$

Performance Metrics for Parallel Systems: Speedup

- What is the benefit from parallelism?
- **Speedup** (***S***) is the ratio of the time taken to solve a problem on a single processor to the time required to solve the same problem on a parallel computer with ***p*** identical processing elements.

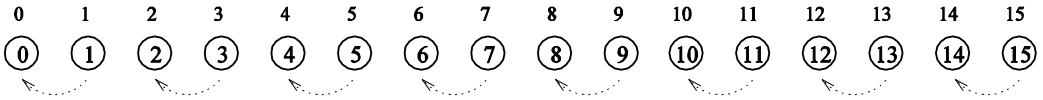
$$S = \frac{T_S}{T_P}$$

Performance Metrics: Example

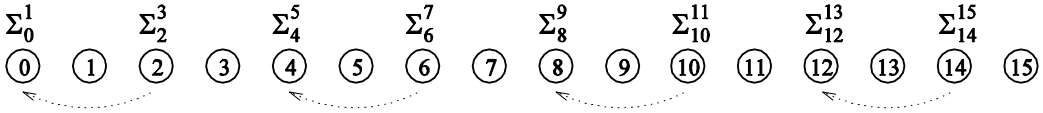
- Consider the problem of **adding *n* numbers** by using ***n*** processing elements.

- If ***n*** is a power of two, we can perform this operation in **log *n*** steps by **propagating partial sums** up a logical binary tree of processors.

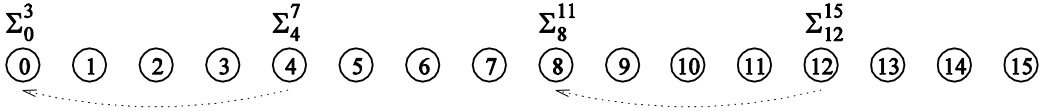
Performance Metrics: Example



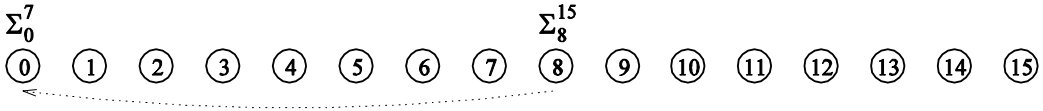
(a) Initial data distribution and the first communication step



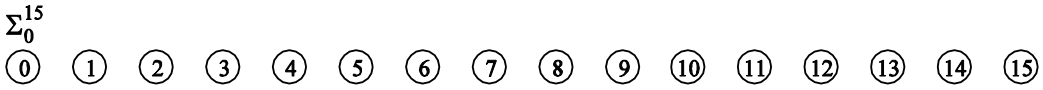
(b) Second communication step



(c) Third communication step



(d) Fourth communication step



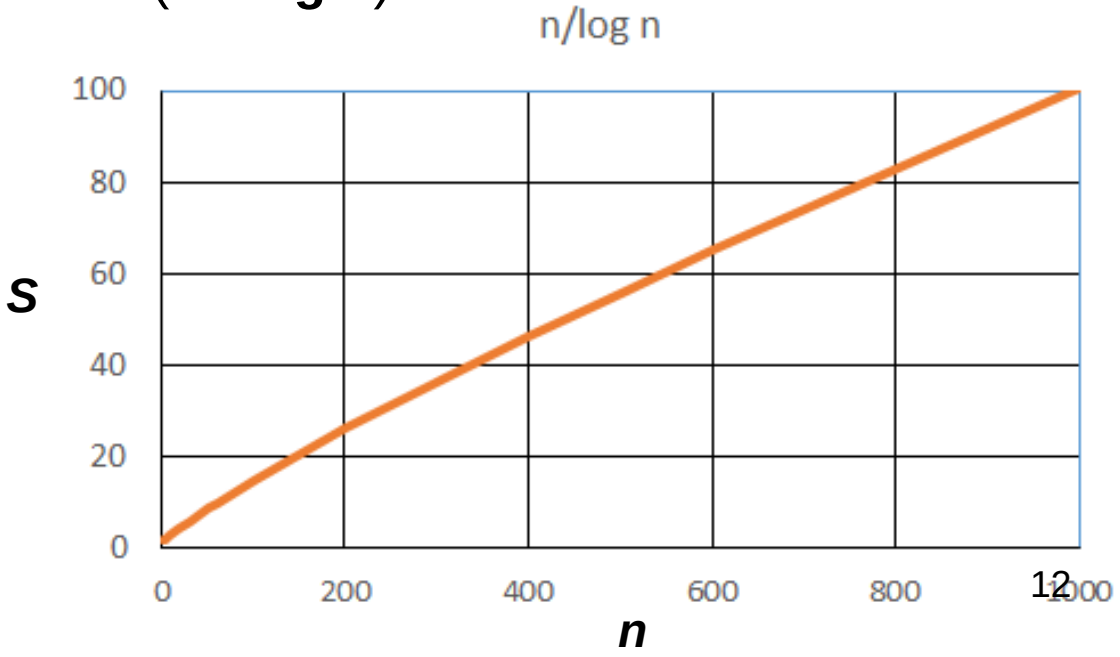
(e) Accumulation of the sum at processing element 0 after the final communicatior

Performance Metrics: Example (continued)

- If an addition takes **constant time**, say, t_c and communication of a single word takes time $t_s + t_w$, we have the parallel time $T_P = \Theta$ (**log** n)

- We know that $T_S = \Theta$ (n)

- Speedup S is given by $S = \Theta$ (n / **log** n)



Performance Metrics: Speedup

- For a given problem, there might be **many serial algorithms** available. These algorithms may have different asymptotic runtimes and may be parallelizable to different degrees.

- For the purpose of computing speedup, we always consider **the best sequential program as the baseline**.

Performance Metrics: Speedup Example

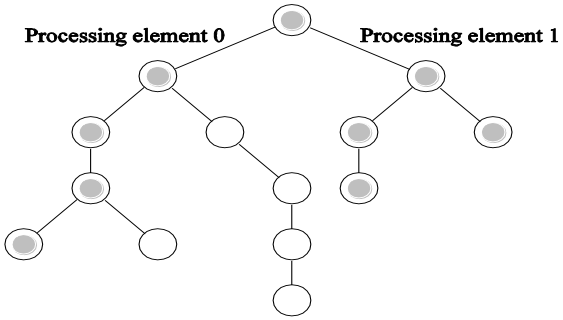
- Consider the problem of **parallel bubble sort**.
- The **serial time** for bubblesort is **150** seconds.
- The parallel time for **odd-even sort** (efficient parallelization of bubble sort) is **40** seconds.
- The speedup would appear to be 150/40 = **3.75**.
- But is this really a fair assessment of the system?
- What if **serial quicksort** only took 30 seconds? In this case, the speedup is 30/40 = 0.75. This is a more realistic assessment of the system.

Performance Metrics: Speedup Bounds

- **Speedup can be as low as 0** (the parallel program never terminates).
- **Speedup, in theory, should be upper bounded by *p*** - after all, we can only expect a ***p***-fold speedup if we use times as many resources.
- A **speedup greater than *p* is possible** only if each processing element spends less than time T_S / p solving the problem.
- In this case, a single processor could be timeslided to achieve a faster serial program, which contradicts our assumption of fastest serial program as basis for speedup.

Performance Metrics: Superlinear Speedups

One reason for **superlinearity** is that the parallel version does less work than corresponding serial algorithm.



Searching an unstructured tree for a node with a given label, `S', on two processing elements using **depth-first traversal**. The **two-processor** version with processor 0 searching the left subtree and processor 1 searching the right subtree expands only the shaded nodes before the solution is found. The corresponding **serial formulation expands the entire tree**. It is clear that **the serial algorithm does more work than the parallel algorithm**.

Amdahl's Law

- All programs contain parts that are naturally sequential (β) and the other fraction is naturally parallel (1 - β).

- Speedup of the algorithm is limited by the naturally sequential part of the algorithm. Amdahl's Law defines theoretically possible speedup, ignoring overhead and communication costs.

- The serial part of the program can be computed in $\beta \boldsymbol{T_S}$, and the parallel program in time (1- β)$\boldsymbol{T_S}$/**p**. Then $\boldsymbol{T_P} = \beta \boldsymbol{T_S} + (1-\beta)\boldsymbol{T_S}/\mathbf{p}$.

- The speedup is then $S \leq \frac{T_S}{\beta T_S + (1-\beta)T_S/p} = \frac{p}{\beta p + (1-\beta)}$

- Example: β = 0.1, p=100 => S ≤ 9.1
  β = 0.1, p=1000 => S ≤ 9.91

Performance Metrics: Efficiency

- **Efficiency** is a measure of the **fraction of time for which a processing element is usefully employed**

- Mathematically, it is given by

$$E = \frac{S}{p}. \qquad (2)$$

- Following the bounds on speedup, efficiency can be **as low as 0 and as high as 1**.

Performance Metrics: Efficiency Example

- The speedup of **adding numbers** on processors is given by

$$S = \frac{n}{\log n}$$

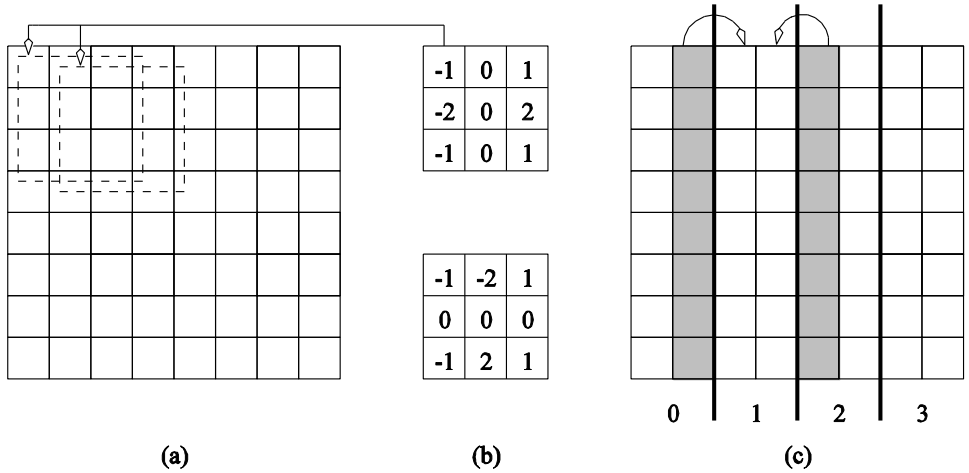
- Efficiency is given by

$$E = \frac{\Theta\left(\frac{n}{\log n}\right)}{n}$$

$$= \Theta\left(\frac{1}{\log n}\right)$$

Parallel Time, Speedup, and Efficiency Example

Consider the problem of **edge-detection in images**. The problem requires us to apply a ***3*** x ***3*** **template** to each pixel. If each multiply-add operation takes time t_c, the serial time for an ***n*** x ***n*** image is given by $T_S = 9t_c n^2$.



Example of edge detection: (a) an ***8*** x ***8*** image; (b) typical templates for detecting edges; and (c) partitioning of the image across **four processors** with **shaded regions indicating image data that must be communicated** from neighboring processors to processor 1.

Parallel Time, Speedup, and Efficiency Example (continued)

- One possible parallelization partitions the image equally into vertical segments, each with n^2 / p pixels.

- The boundary of each segment is $2n$ pixels. This is also the number of pixel values that will have to be communicated. This takes time $2(t_s + t_w n)$.

- Templates may now be applied to all n^2 / p pixels in time $9 t_c n^2 / p$.

Parallel Time, Speedup, and Efficiency Example (continued)

- The total time for the algorithm is therefore given by:

$$T_P = 9t_c\frac{n^2}{p} + 2(t_s + t_w n)$$

- The corresponding values of speedup and efficiency are given by:

$$S = \frac{9t_c n^2}{9t_c\frac{n^2}{p} + 2(t_s + t_w n)}$$

and

$$E = \frac{1}{1 + \frac{2p(t_s + t_w n)}{9t_c n^2}}.$$

Cost of a Parallel System

- **Cost** is the product of parallel runtime and the number of processing elements used ($\boldsymbol{p}$ x $\boldsymbol{T_P}$).
- Cost reflects the sum of the **time** that each processing element **spends solving the problem**.
- A parallel system is said to be ***cost-optimal*** if **the cost of solving a problem on a parallel computer is asymptotically identical to serial cost**.
- Since $\boldsymbol{E} = \boldsymbol{T_S} / \boldsymbol{p}\ \boldsymbol{T_P}$, for cost optimal systems, $\boldsymbol{E} = O(1)$.
- Cost is sometimes referred to as *work* or *processor-time product*.

Cost of a Parallel System: Example

Consider the problem of **adding numbers** on processors.

- We have, T_P = **log** n (for p = n).
- The cost of this system is given by $p\ T_P$ = n **log** n.
- Since the serial runtime of this operation is $\Theta(n)$, the algorithm is **not cost optimal**.

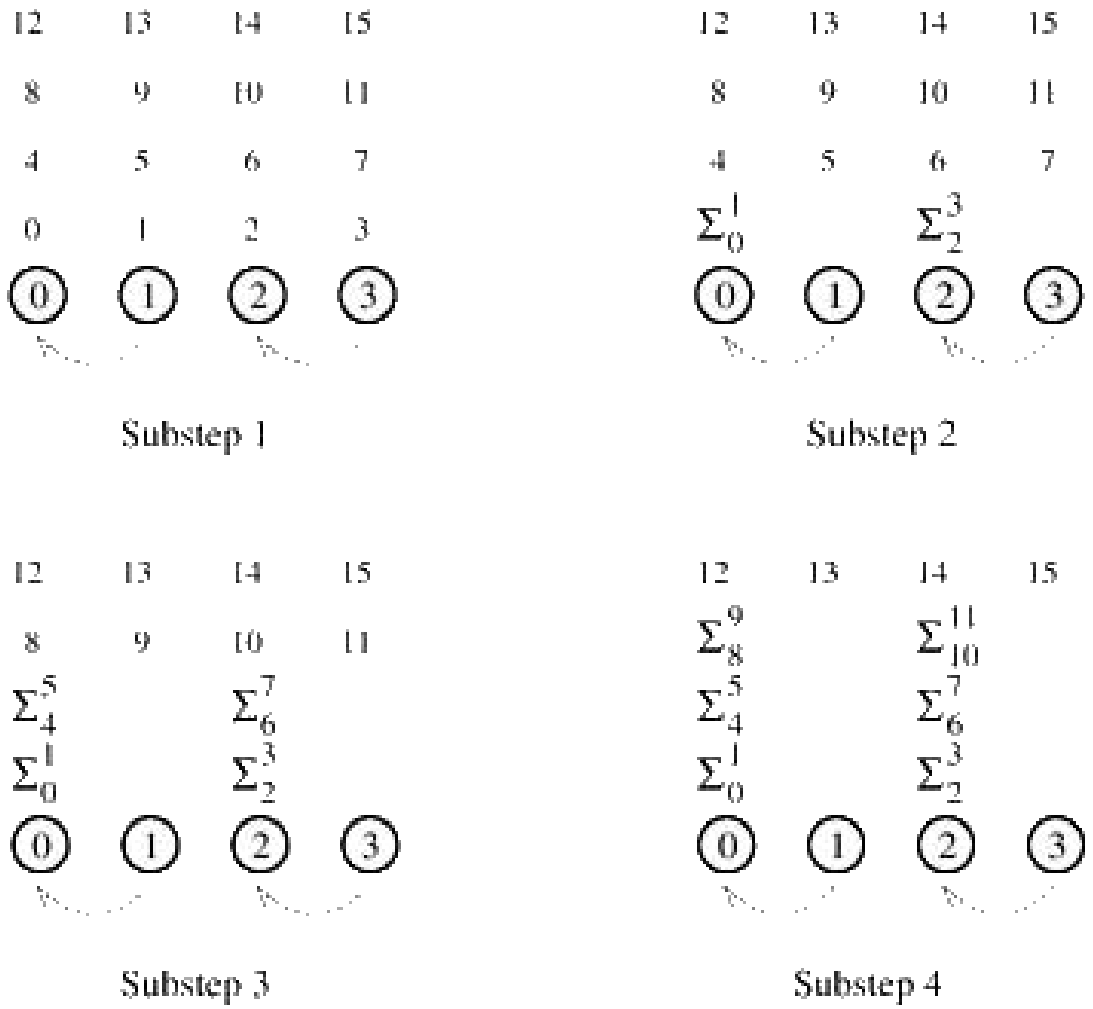
Effect of Granularity on Performance

- **Often, using fewer processors improves performance** of parallel systems.
- Using fewer than the maximum possible number of processing elements to execute a parallel algorithm is called ***scaling down*** a parallel system.
- A **naive way of scaling down** is to think of each processor in the original case as a **virtual processor** and to assign virtual processors equally to scaled down processors.
- Since the number of processing elements decreases by a factor of ***n*** / ***p***, the computation at each processing element increases by a factor of ***n*** / ***p***.
- The communication cost should not increase by this factor since **some of the virtual processors assigned to a physical processors might talk to each other**. This is the basic reason for the improvement from building granularity.

Building Granularity: Example

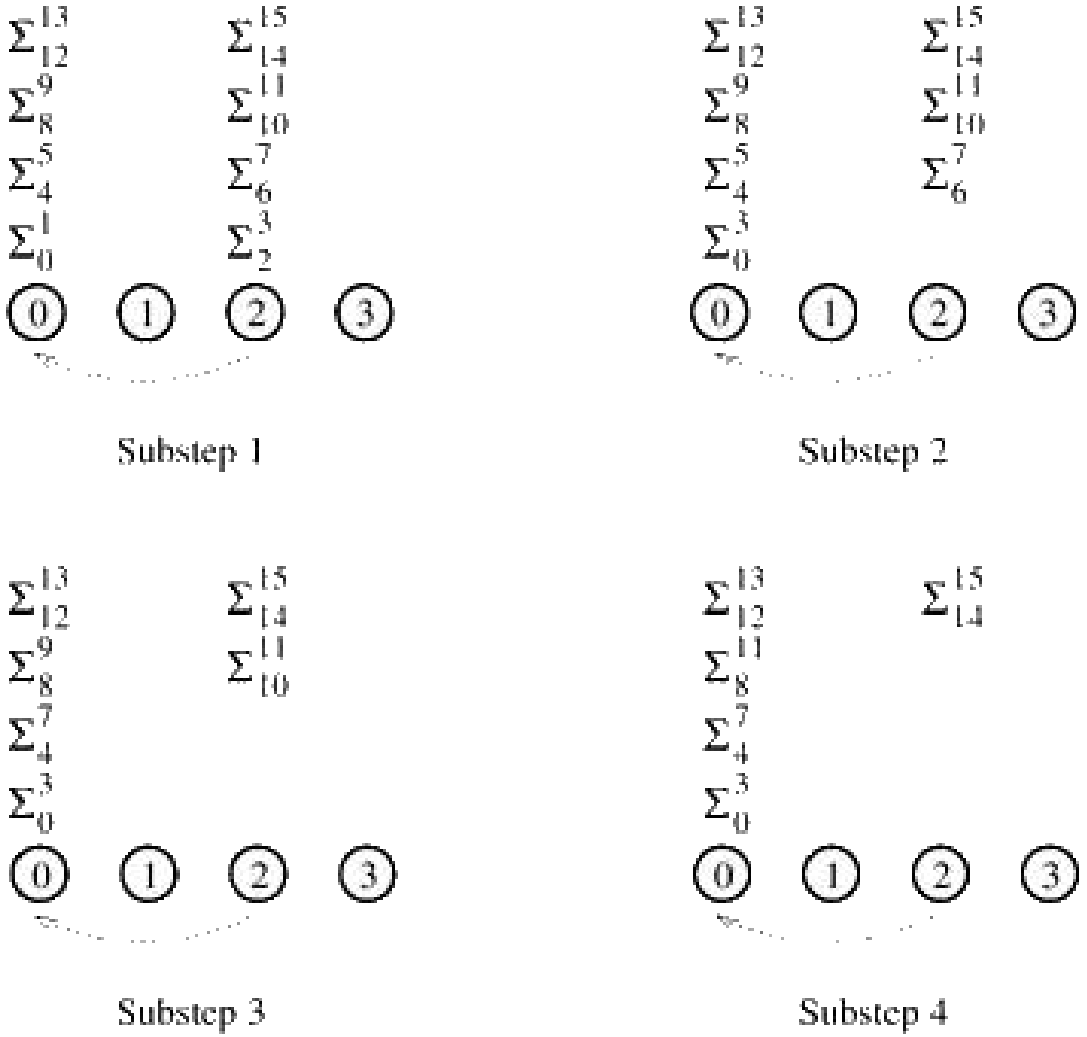
- Consider the problem of **adding *n* numbers** on ***p*** processing elements such that ***p*** < ***n*** and both ***n*** and ***p*** are powers of 2.
- Use the parallel algorithm for ***n*** processors, except, in this case, we think of them as **virtual processors**.
- Each of the ***p*** processors is now assigned ***n*** / ***p*** virtual processors.
- The first **log *p*** of the **log *n*** steps of the original algorithm are simulated in (***n*** / ***p***) **log *p*** steps on ***p*** processing elements.
- Subsequent **log *n* - log *p*** steps **do not require any communication**.

Building Granularity: Example (continued)



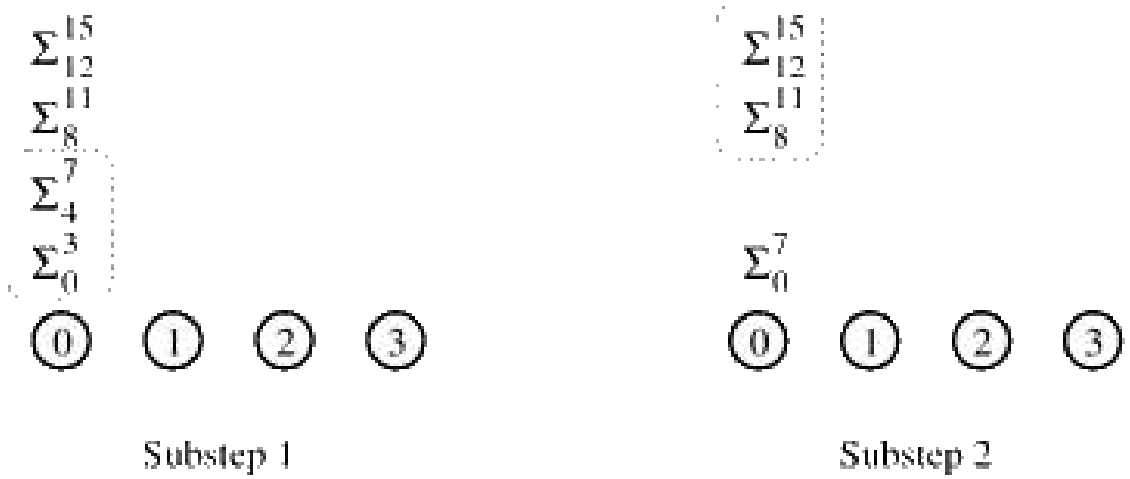
(a) Four processors simulating the first communication step of 16 processors

Building Granularity: Example (continued)



(b) Four processors simulating the second communication step of 16 processors

Building Granularity: Example (continued)



(c) Simulation of the third step in two substeps

(d) Simulation of the fourth step

(e) Final result

Building Granularity: Example (continued)

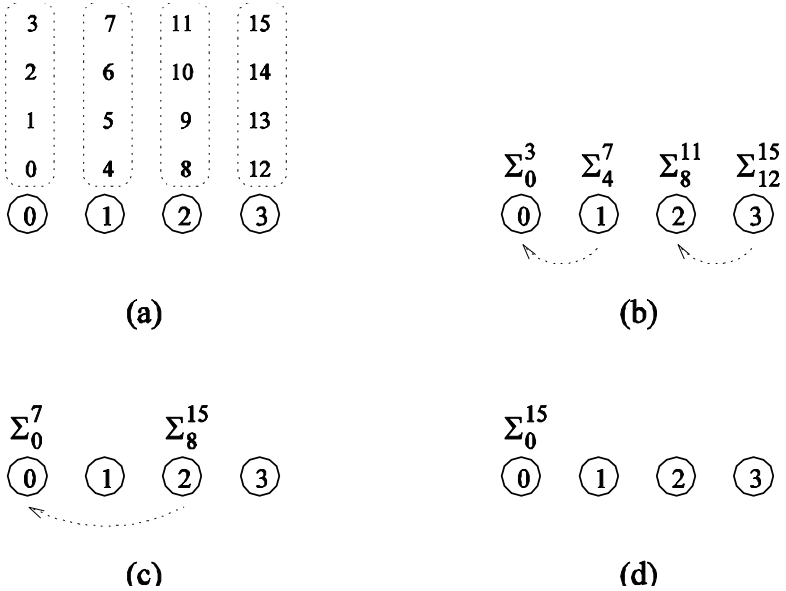
- The overall parallel execution time of this parallel system is $\Theta\left(\left(\boldsymbol{n} / \boldsymbol{p}\right) \textbf{log } \boldsymbol{p}\right)$.

- The cost is $\Theta\left(\boldsymbol{n} \textbf{ log } \boldsymbol{p}\right)$, which is asymptotically higher than the $\Theta\left(\boldsymbol{n}\right)$ cost of adding $\boldsymbol{n}$ numbers sequentially. Therefore, the parallel system is **not cost-optimal**.

Building Granularity: Example (continued)

Can we build granularity in the example in a cost-optimal fashion?

- Each processing element locally adds its ***n*** / ***p*** numbers in time Θ (***n*** / ***p***).
- The ***p*** partial sums on ***p*** processing elements can be added in time Θ(**log** ***p***).



A cost-optimal way of computing the sum of 16 numbers using four processing elements.

Building Granularity: Example (continued)

- The parallel runtime of this algorithm is

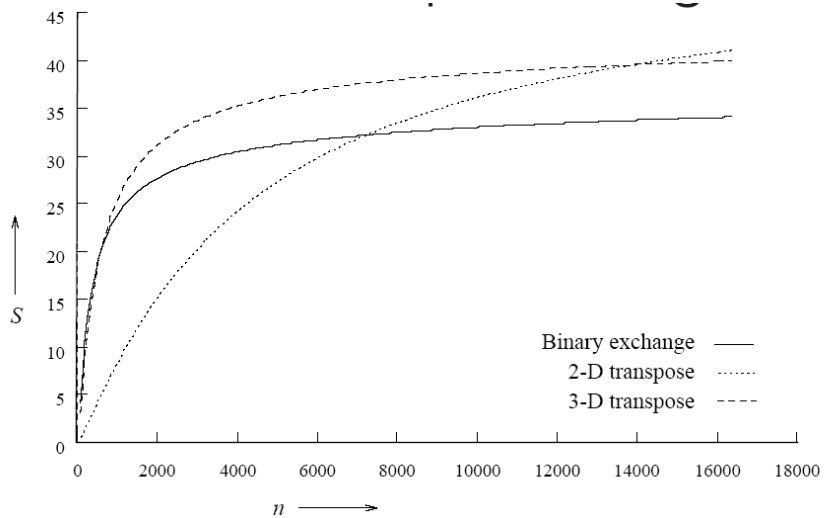
$$T_P = \Theta(n/p + \log p), \tag{3}$$

- The cost is $\Theta(n + p \log p)$
- This is **cost-optimal**, so long as $n = \Omega(p \log p)$!

Scalability of Parallel Systems

How do we extrapolate performance from small problems and small systems to larger problems on larger configurations?

Consider three parallel algorithms for computing an ***n***-point Fast Fourier Transform (FFT) on 64 processing elements.



A comparison of the speedups obtained by the binary-exchange, 2-D transpose and 3-D transpose algorithms on 64 processing elements with $t_c = 2$, $t_w = 4$, $t_s = 25$, and $t_h = 2$.

Clearly, it is difficult to infer scaling characteristics from observations on small datasets on small machines.

Scaling Characteristics of Parallel Programs

- The efficiency of a parallel program can be written as:

$$E = \frac{S}{p} = \frac{T_S}{pT_P}$$

or

$$E = \frac{1}{1 + \frac{T_o}{T_S}}. \qquad (4)$$

- The total overhead function T_o is an increasing function of p.

- For a given problem size (i.e., the value of T_S **remains constant**), **as we increase the number of processing elements, T_o increases**.

- The overall **efficiency of the parallel program goes down**. This is the case for all parallel programs.

Scaling Characteristics of Parallel Programs: Example

- Consider the problem of adding numbers on processing elements.
- We have seen that:

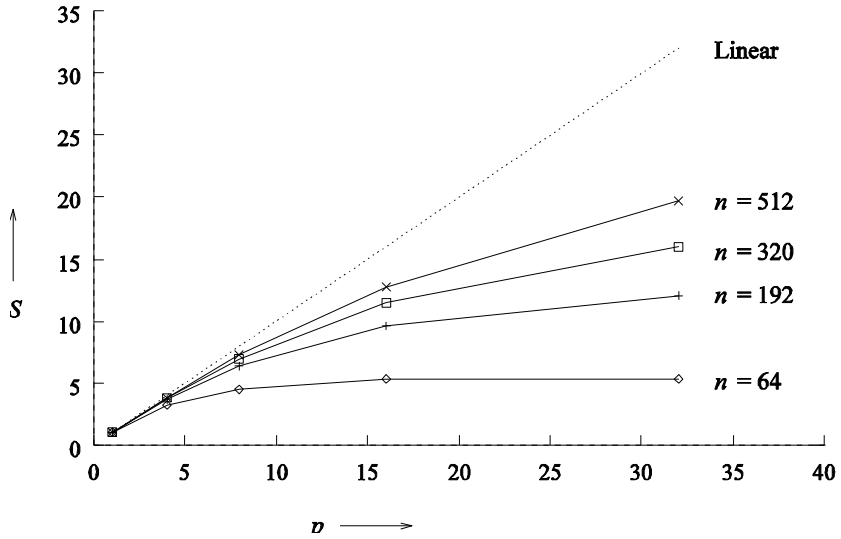
$$T_P = \frac{n}{p} + 2\log p \quad (5)$$

$$S = \frac{n}{\frac{n}{p} + 2\log p} \quad (6)$$

$$E = \frac{1}{1 + \frac{2p\log p}{n}} \quad (7)$$

Scaling Characteristics of Parallel Programs: Example (continued)

Plotting the speedup for various input sizes gives us:



Speedup versus the **number of processing elements** for adding a list of numbers.

Speedup tends to saturate and **efficiency drops** as a consequence of Amdahl's law

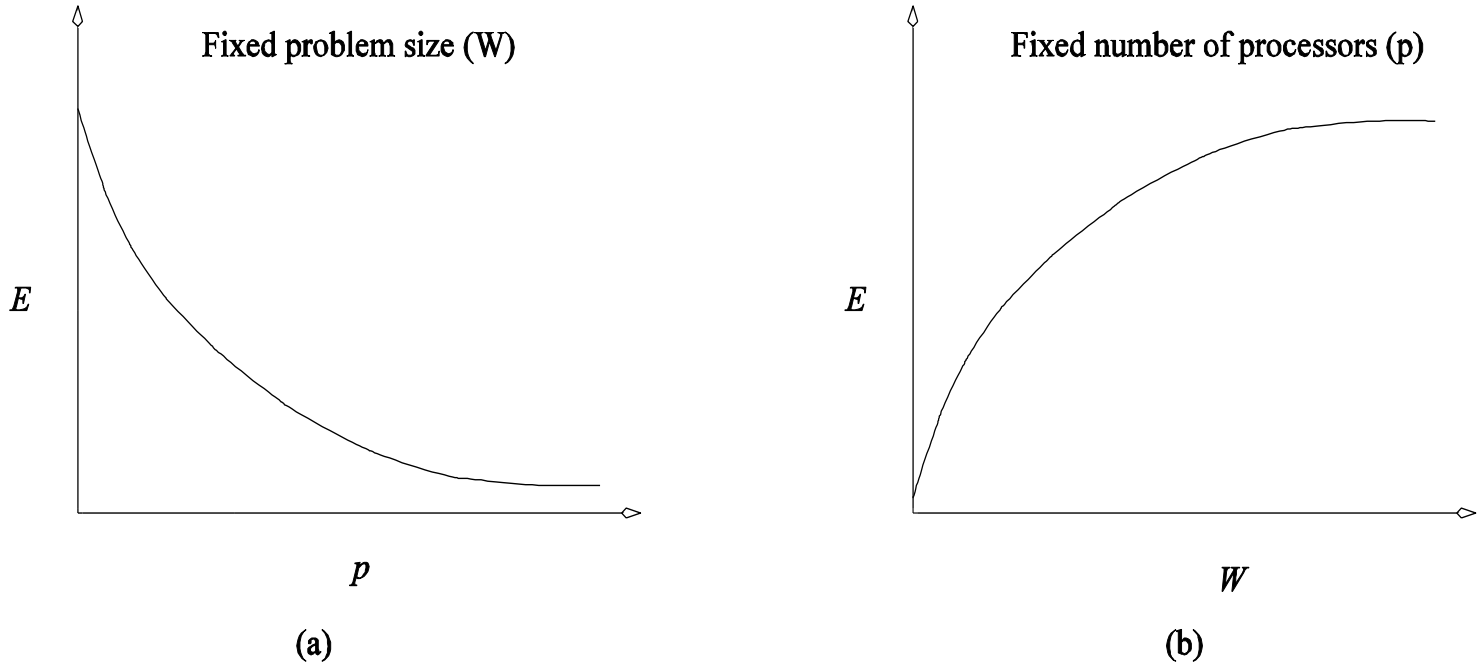
Scaling Characteristics of Parallel Programs

- Total overhead function T_o is a **function of both problem size** (n -> T_s) and the **number of processing elements *p***.
- In many cases, T_o grows sublinearly with respect to T_s.
- In such cases, the **efficiency increases if the problem size is increased keeping the number of processing elements constant**.
- For such systems, we can **simultaneously increase the problem size and number of processors to keep efficiency constant**.
- We call such systems ***scalable* parallel systems**.

Scaling Characteristics of Parallel Programs

- Recall that **cost-optimal parallel systems have an efficiency of Θ(1)**.
- **Scalability** and **cost-optimality** are therefore related.
- **A scalable parallel system can always be made cost-optimal** if the number of processing elements and the size of the computation are chosen appropriately.
- For a **given problem size, as we increase the number of processing elements**, the overall efficiency of the parallel system goes down for all systems.

Isoefficiency Metric of Scalability



Variation of efficiency: (a) as the number of processing elements is increased for a given problem size; and (b) as the problem size is increased for a given number of processing elements. The phenomenon illustrated in graph (b) is not common to all parallel systems.

Isoefficiency Metric of Scalability

- What is the **rate at which the problem size must increase with respect to the number of processing elements** to keep the efficiency fixed?

- This rate **determines the scalability** of the system. **The slower** this rate, **the better**.

- Before we formalize this rate, we define the problem size ***W*** as the asymptotic **number of operations associated with the best serial algorithm** to solve the problem.

Isoefficiency Metric of Scalability

- We can write parallel runtime as:

$$T_P = \frac{W + T_o(W,p)}{p} \tag{8}$$

- The resulting expression for speedup is

$$\begin{aligned} S &= \frac{W}{T_P} \\ &= \frac{Wp}{W + T_o(W,p)}. \end{aligned} \tag{9}$$

- Finally, we write the expression for efficiency as

$$\begin{aligned} E &= \frac{S}{p} \\ &= \frac{W}{W + T_o(W,p)} \\ &= \frac{1}{1 + T_o(W,p)/W}. \end{aligned}$$

Isoefficiency Metric of Scalability

- For **scalable parallel systems, efficiency can be maintained** at a fixed value (between 0 and 1) if the ratio T_o / W is maintained at a constant value.
- For a desired value E of efficiency,

$$\begin{aligned} E &= \frac{1}{1 + T_o(W,p)/W}, \\ \frac{T_o(W,p)}{W} &= \frac{1-E}{E}, \\ W &= \frac{E}{1-E} T_o(W,p). \end{aligned} \tag{11}$$

- If $K = E / (1 - E)$ is a **constant depending on the efficiency** to be maintained, since T_o is a function of W and p, we have

$$W = K T_o(W,p). \tag{12}$$

Isoefficiency Metric of Scalability

- The problem size ***W*** can usually be **obtained as a function** of ***p*** by **algebraic manipulations** to keep efficiency constant.

- This function is called the *isoefficiency function*.

- This function determines the ease with which a parallel system can **maintain a constant efficiency** and hence **achieve speedups increasing in proportion to the number of processing elements.**

- If *W* needs to grow only linearly with respect to *p*, then the parallel system is **highly scalable**. On the other hand, if *W* might need to grow, e.g., as an exponential function of *p* to keep the efficiency from dropping as p increases. Such parallel systems are **poorly scalable**.

Isoefficiency Metric: Example

- The overhead function for the problem of adding ***n*** numbers on ***p*** processing elements is approximately ***2p* log *p*** .

- Substituting T_o by ***2p* log *p*** , we get

$$W = K2p\log p. \quad (13)$$

Thus, the asymptotic isoefficiency function for this parallel system is $\Theta(p\log p)$.

- If the number of processing elements is increased from ***p*** to ***p'***, the problem size (in this case, n) must be increased by a factor of (***p'* log *p'***) / (***p* log *p***) to get the same efficiency as on ***p*** processing elements.

Isoefficiency Metric: Example

Consider a more complex example where $T_o = p^{3/2} + p^{3/4}W^{3/4}$

- Using only the first term of T_o in Equation 12, we get

$$W = Kp^{3/2}. \quad (14)$$

- Using only the second term, Equation 12 yields the following relation between W and p:

$$\begin{aligned} W &= Kp^{3/4}W^{3/4} \\ W^{1/4} &= Kp^{3/4} \\ W &= K^4p^3 \end{aligned} \quad (15)$$

- The larger of these two asymptotic rates determines the isoefficiency. This is given by $\Theta(p^3)$

Cost-Optimality and the Isoefficiency Function

- A parallel system is cost-optimal if and only if

$$pT_P = \Theta(W). \tag{16}$$

- From this, we have:

$$W + T_o(W,p) = \Theta(W)$$

$$T_o(W,p) = O(W) \tag{17}$$

$$W = \Omega(T_o(W,p)) \tag{18}$$

- If we have an isoefficiency function ***f***(***p***), then it follows that the relation ***W*** = Ω(***f***(***p***)) must be satisfied to ensure the cost-optimality of a parallel system as it is scaled up.

Lower Bound on the Isoefficiency Function

- For a problem consisting of ***W*** units of work, **no more than *W* processing elements can be used cost-optimally**.

- The problem size must increase at least as fast as Θ(***p***) to maintain fixed efficiency; hence, Ω(***p***) is the asymptotic lower bound on the isoefficiency function.

Degree of Concurrency and the Isoefficiency Function

- The maximum number of tasks that can be executed simultaneously at any time in a parallel algorithm is called its *degree of concurrency*.

- If $C(W)$ is the degree of concurrency of a parallel algorithm, then for a problem of size W, no more than $C(W)$ processing elements can be employed effectively.

Degree of Concurrency and the Isoefficiency Function: Example

Consider solving a system of equations in variables by using Gaussian elimination ($W = \Theta(n^3)$)

- The n variables must be eliminated one after the other, and eliminating each variable requires $\Theta(n^2)$ computations.
- At most $\Theta(n^2)$ processing elements can be kept busy at any time.
- Since $W = \Theta(n^3)$ for this problem, the degree of concurrency $C(W)$ is $\Theta(W^{2/3})$.
- Given p processing elements, the problem size should be at least $\Omega(p^{3/2})$ to use them all.

Minimum Execution Time

Often, we are interested in **the minimum time to solution**.

- We can determine the minimum parallel runtime T_P^{min} for a given W by differentiating the expression for T_P w.r.t. p and equating it to zero.

$$\frac{\mathrm{d}}{\mathrm{d}p} T_P = 0 \qquad (19)$$

- If p_0 is the value of p as determined by this equation, $T_P(p_0)$ is the minimum parallel time.

Minimum Execution Time: Example

Consider the minimum execution time for adding ***n*** numbers.

$$T_P = \frac{n}{p} + 2\log p. \quad (20)$$

Setting the derivative w.r.t. ***p*** to zero, we have ***p = n/ 2*** . The corresponding runtime is

$$T_P^{min} = 2\log n. \quad (21)$$

(One may verify that this is indeed a min by verifying that the second derivative is positive).

Note that at this point, the formulation is **not cost-optimal**.

Asymptotic Analysis of Parallel Programs

Consider the problem of sorting a list of ***n*** numbers. The fastest serial programs for this problem run in time Θ(***n* log *n***). Consider four parallel algorithms, A1, A2, A3, and A4 as follows:

Comparison of four different algorithms for sorting a given list of numbers. The table shows number of processing elements, parallel runtime, speedup, efficiency and the pT_P product.

Algorithm	A1	A2	A3	A4
p	n^2	$\log n$	n	$\sqrt{n}$
T_P	1	n	$\sqrt{n}$	$\sqrt{n}\log n$
S	$n\log n$	$\log n$	$\sqrt{n}\log n$	$\sqrt{n}$
E	$\frac{\log n}{n}$	1	$\frac{\log n}{\sqrt{n}}$	1
pT_P	n^2	$n\log n$	$n^{1.5}$	$n\log n$

Asymptotic Analysis of Parallel Programs

- If the metric is **speed**, algorithm A1 is the best, followed by A3, A4, and A2 (in order of increasing T_P).
- In terms of **efficiency**, A2 and A4 are the best, followed by A3 and A1.
- In terms of **cost**, algorithms A2 and A4 are cost optimal, A1 and A3 are not.
- It is important to identify the objectives of analysis and to **use appropriate metrics**!