Search Algorithms for Discrete Optimization Problems

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Topic Overview

- Discrete Optimization - Basics
- Sequential Search Algorithms
- Parallel Depth-First Search
- Parallel Best-First Search
- Speedup Anomalies in Parallel Search Algorithms

Discrete Optimization - Basics

- Discrete optimization forms a class of **computationally expensive problems** of significant theoretical and practical interest.
- Search algorithms **systematically search the space** of possible solutions subject to constraints.

Definitions

- A *discrete optimization problem* can be expressed as a tuple (S, f). The set S is a finite or countably infinite **set of all solutions** that satisfy specified constraints.
- The function f is the **cost function** that maps each element in set S onto the set of real numbers R.
- The objective of a DOP is to find a **feasible solution** x_{opt}, such that $f(x_{opt}) \leq f(x)$ for all $x \in S$.
- A number of diverse problems such as VLSI layouts, **robot motion planning**, test pattern generation, and **facility location** can be formulated as DOPs.

Discrete Optimization: Example

- In the 0/1 integer-linear-programming problem, we are given an $m \times n$ matrix A, an $m \times 1$ vector b, and an $n \times 1$ vector c.
- The objective is to determine an $n \times 1$ vector $\overline{x}$ whose elements can take on only the value 0 or 1.
- The vector must satisfy the constraint

$$A\overline{x} \geq b$$

and the function

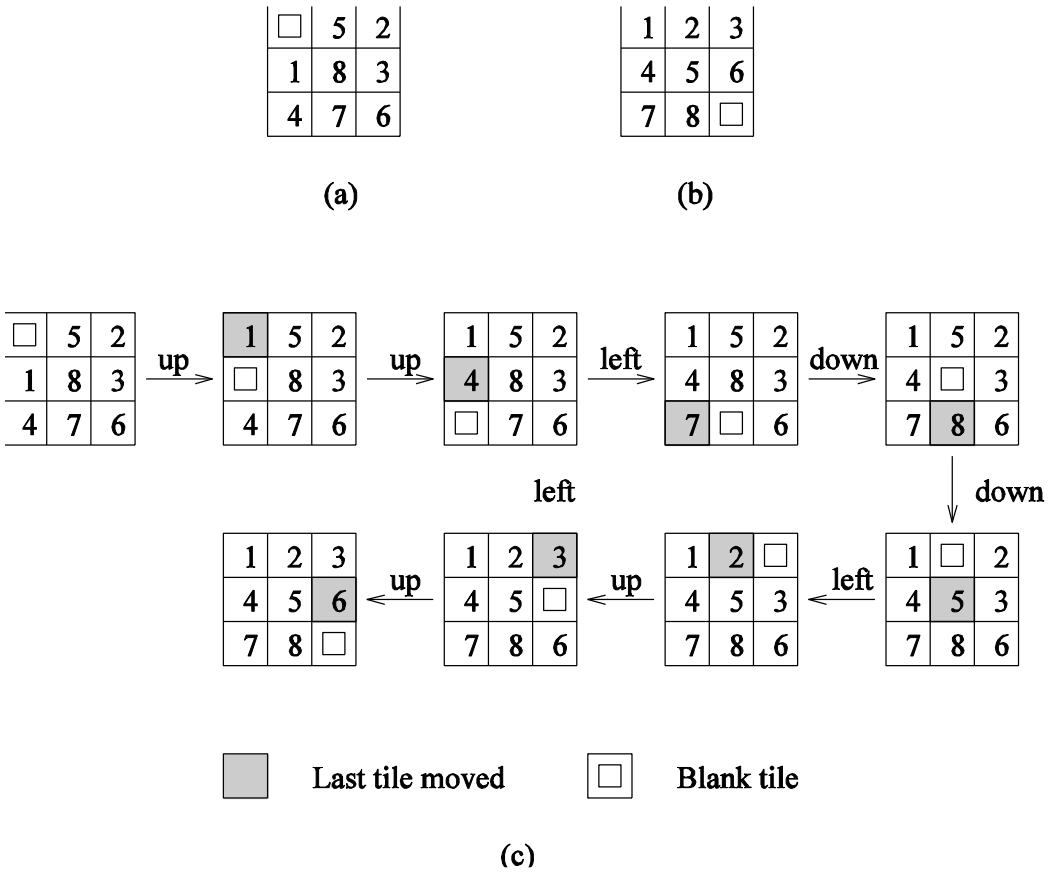
$$f(\overline{x}) = c^T \overline{x}$$

must be minimized.

Discrete Optimization: Example

- The **8-puzzle** problem consists of a 3×3 grid containing eight tiles, numbered one through eight.
- One of the grid segments (called the ``**blank**'') is empty. A tile can be moved into the blank position from a position adjacent to it, thus creating a blank in the tile's original position.
- The goal is to move from a given initial position to the final position in a **minimum number of moves**.

Discrete Optimization: Example



An 8-puzzle problem instance: (a) initial configuration; (b) final configuration; and (c) a sequence of moves leading from the initial to the final configuration.

Discrete Optimization Basics

- The feasible space S is typically **very large**.
- For this reason, a DOP can be reformulated as the problem of **finding a minimum-cost path in a graph** from a designated initial node to one of several possible goal nodes.
- Each element x in S can be viewed as a path from the initial node to one of the goal nodes.
- This graph is called a ***state space***.

Discrete Optimization Basics

- Often, it is possible to **estimate the cost** to reach the goal state from an intermediate state.
- This estimate, called a ***heuristic estimate***, can be effective in guiding search to the solution.
- If the estimate is guaranteed to be an **underestimate**, the heuristic is called an ***admissible heuristic***.
- Admissible heuristics have desirable properties in terms of optimality of solution (as we shall see later).

Discrete Optimization: Example

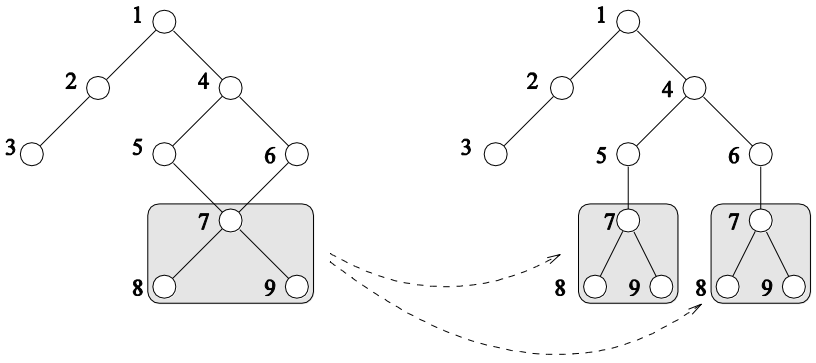
An admissible heuristic for 8-puzzle is as follows:

- Assume that each position in the **8-puzzle grid** is represented as a pair.
- The distance between positions (i,j) and (k,l) is defined as $|i - k| + |j - l|$. This distance is called the **Manhattan distance**.
- The **sum of the Manhattan distances** between the initial and final positions of all tiles is an **admissible heuristic**.

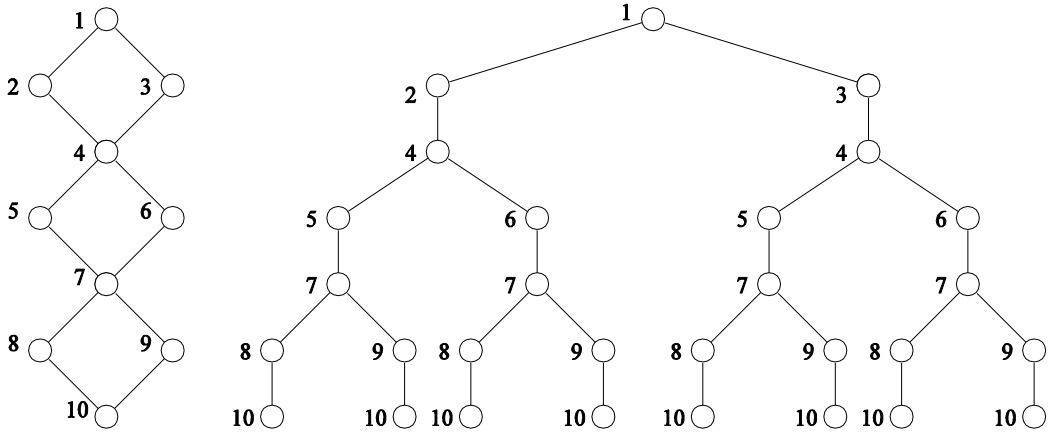
Sequential Search Algorithms

- Is the search space a **tree or a graph**?
- The space of a **0/1 integer program is a tree**, while that of an **8-puzzle is a graph**.
- This has important implications for search since **unfolding a graph into a tree can have significant overheads**.

Sequential Search Algorithms



(a)



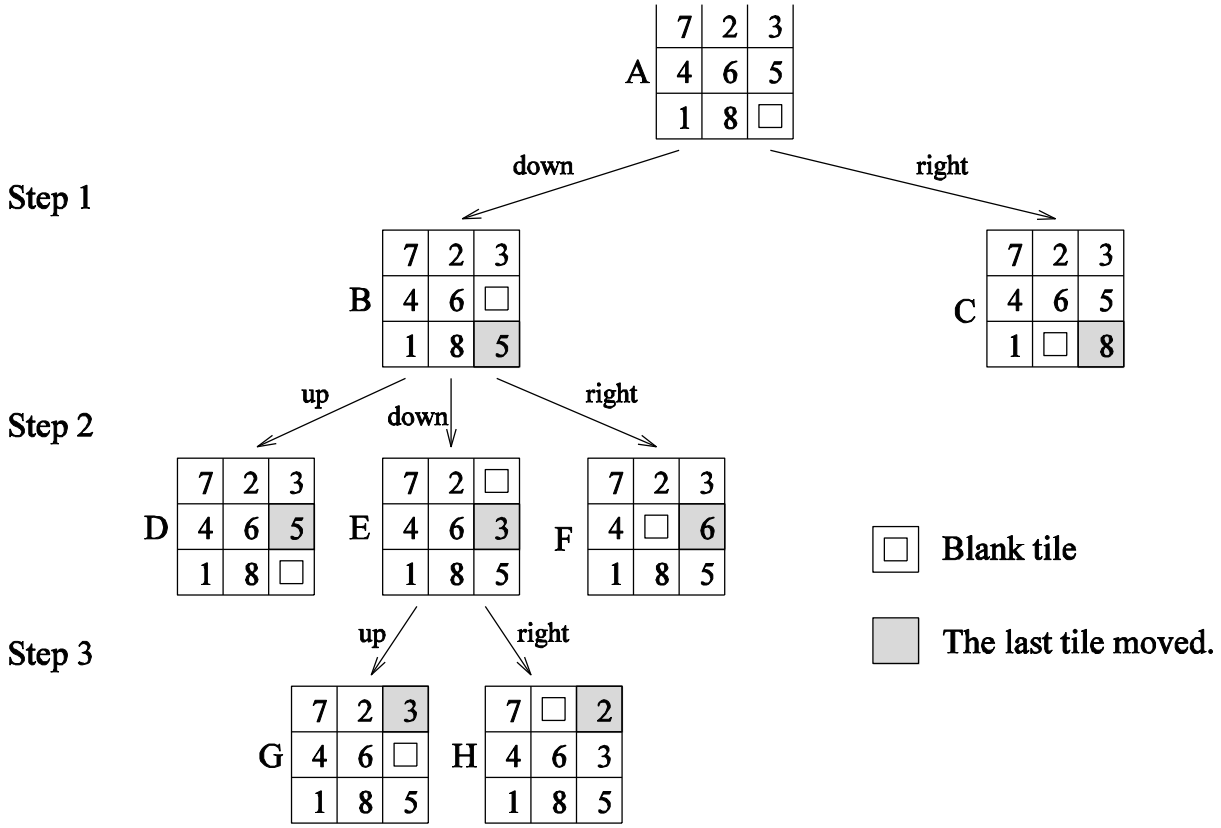
(b)

Two examples of unfolding a graph into a tree.

Depth-First Search Algorithms (DFS)

- Applies to search spaces that are **trees**.
- DFS begins by **expanding the initial node** and **generating its successors**. In each subsequent step, DFS **expands one of the most recently generated nodes**.
- If there exists **no success**, DFS **backtracks** to the parent and explores an alternate child.
- Often, successors of a node are **ordered based on their likelihood of reaching a solution**. This is called **directed DFS**.
- The main advantage of DFS is that its **storage requirement is linear in the depth** of the state space being searched.

Depth-First Search Algorithms



States resulting from **the first three steps** of depth-first search applied to an instance of the 8-puzzle.

DFS Algorithms: Simple Backtracking

- Simple backtracking performs DFS until it finds the **first feasible solution** and terminates.
- Not guaranteed to find a minimum-cost solution.
- **Uses no heuristic** information to order the successors of an expanded node.
- Ordered backtracking uses heuristics to order the successors of an expanded node.

Depth-First Branch-and-Bound (DFBB)

- DFS technique in which upon finding a solution, the algorithm **updates current best solution**.
- DFBB does **not explore paths that are guaranteed to lead to solutions worse than current best solution**.
- On **termination**, the current best solution is a **globally optimal solution**.

Best-First Search (BFS) Algorithms

- BFS algorithms **use a heuristic to guide search**.
- The core data structure is a list, called **Open list**, that **stores unexplored nodes** sorted on their heuristic estimates.
- The **best node is selected** from the list, **expanded**, and its **off-spring are inserted** at the right position.
- If the heuristic is **admissible**, the BFS finds the **optimal** solution.

Best-First Search (BFS) Algorithms

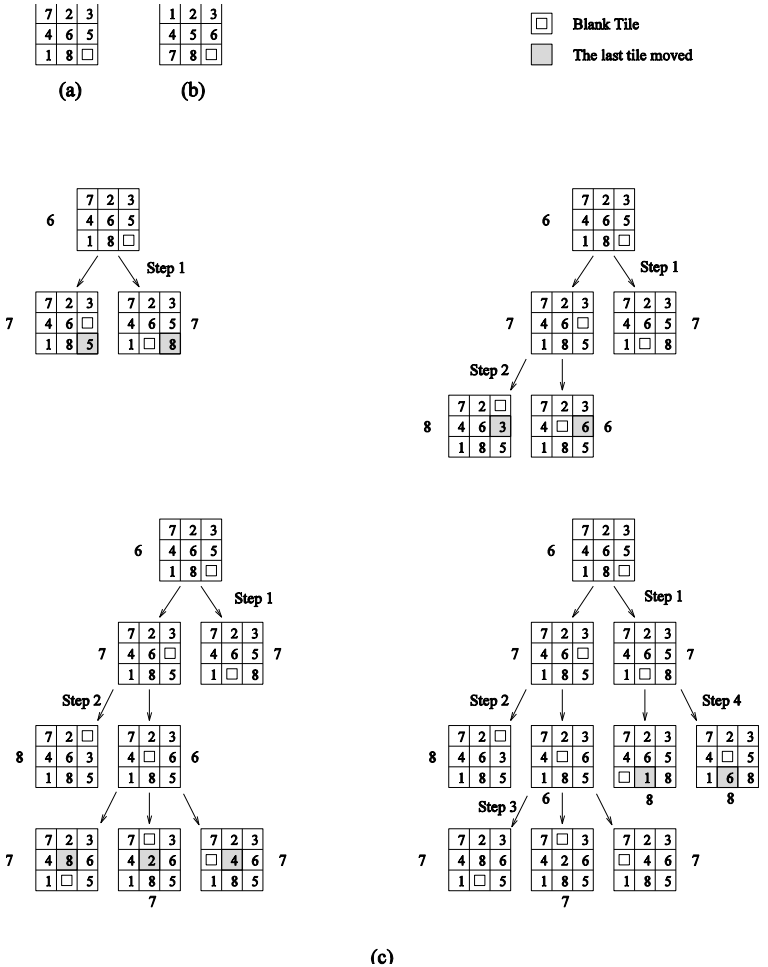
- BFS of graphs must be slightly modified to account for **multiple paths to the same node**.
- A **closed list** stores all the nodes that have been previously seen.
- **If a newly expanded node exists in the open or closed lists with better heuristic value, the node is not inserted into the open list**.

The A* Algorithm

- A BFS technique that uses **admissible heuristics**.
- Defines **function *l*(*x*)** for each node x as $g(x) + h(x)$.
- Here, $g(x)$ is the cost of getting to node x and $h(x)$ is an admissible heuristic estimate of getting from node x to the solution.
- The **open list is sorted** on $l(x)$.

The **space requirement of BFS is exponential in depth**!

Best-First Search: Example

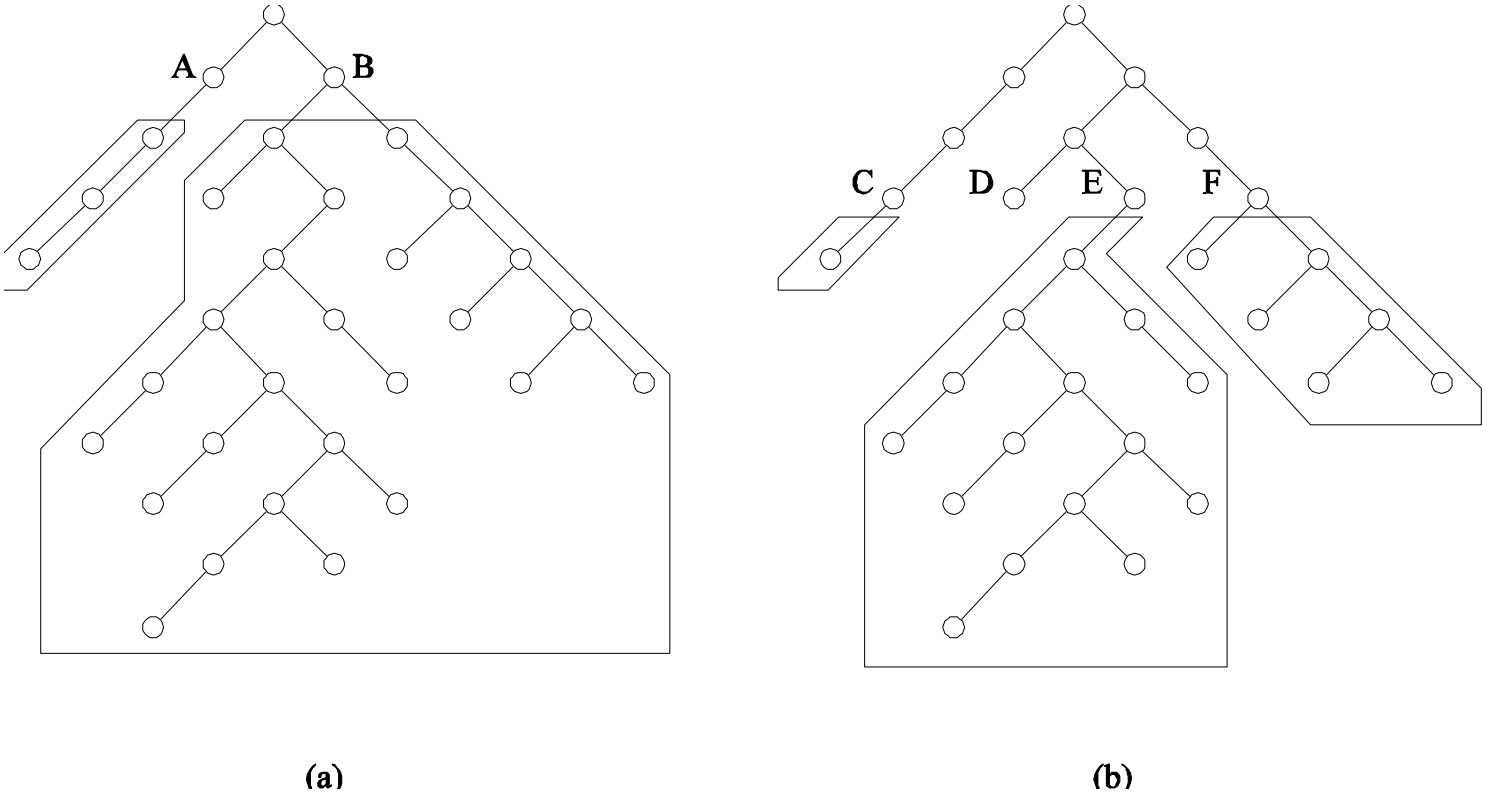


Applying best-first search to the 8-puzzle: (a) initial configuration; (b) final configuration; and (c) states resulting from **the first four steps** of best-first search. Each state is labeled with its -value (that is, the **Manhattan distance** from the state to the final state).

Parallel Depth-First Search

- How is the search space partitioned across processors?
- **Different subtrees can be searched concurrently**.
- However, subtrees can be very **different in size**.
- It is **difficult to estimate the size** of a subtree rooted at a node.
- **Dynamic load balancing** is required.

Parallel Depth-First Search

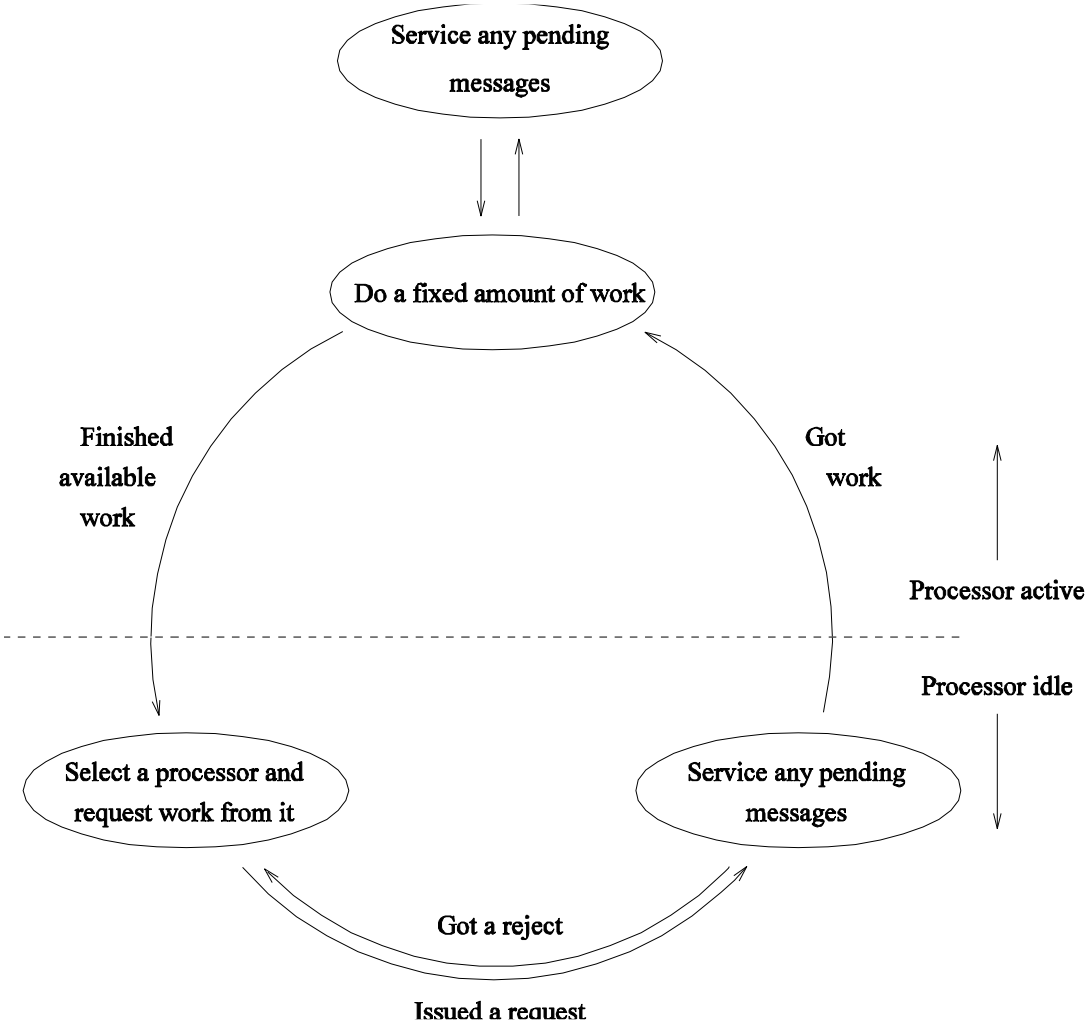


The unstructured nature of tree search and the **imbalance** resulting from static partitioning.

Parallel Depth-First Search: Dynamic Load Balancing

- When a processor **runs out of work**, it **gets more work** from another processor.
- This is done using **work requests and responses** in **message passing machines** and locking and extracting work in shared address space machines.
- On reaching **final state** at a processor, **all processors terminate**.
- Unexplored states can be conveniently stored as **local stacks** at processors.
- The entire space is assigned to one processor to begin with.

Parallel Depth-First Search: Dynamic Load Balancing

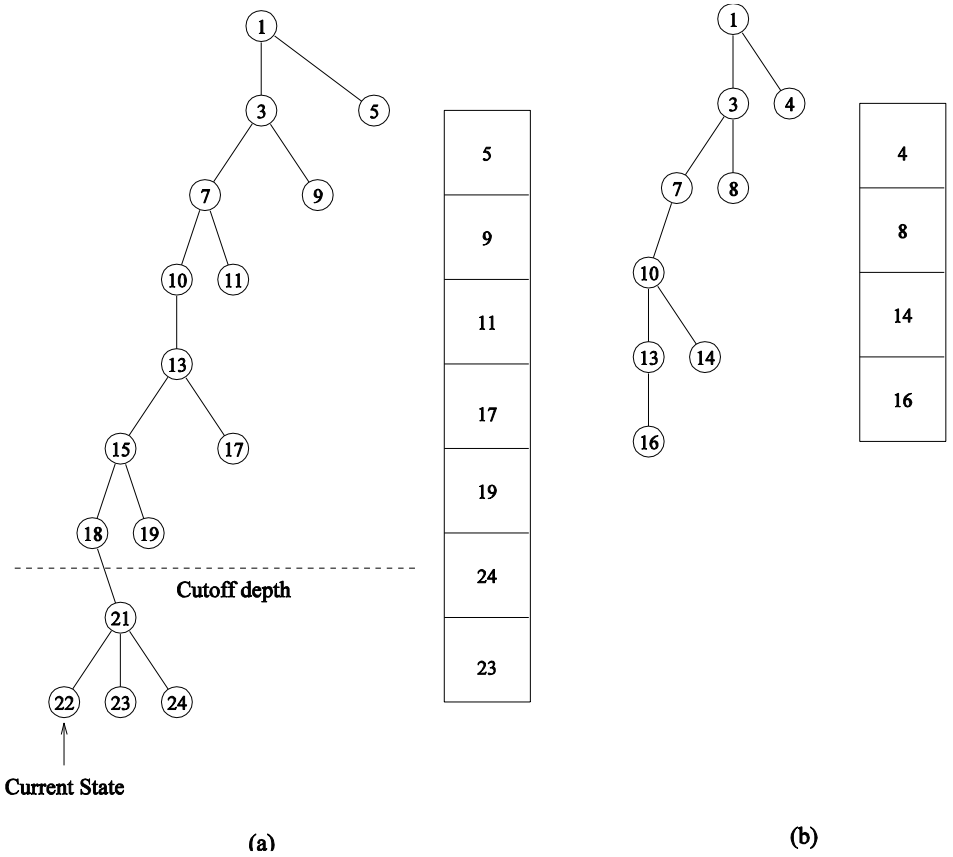


A generic scheme for dynamic load balancing.

Parameters in Parallel DFS: Work Splitting

- Work is split by **splitting the stack into two**.
- Ideally, we do not want either of the **split pieces to be small**.
- **Select nodes near the bottom** of the stack (node splitting), or
- **Select some nodes from each level** (stack splitting).
- The second strategy generally yields a more even split of the space.

Parameters in Parallel DFS: Work Splitting



Splitting the DFS tree: the two subtrees along with their stack representations are shown in (a) and (b).

Load-Balancing Schemes

- **Who do you request work from?** Note that we would like to distribute work requests evenly, in a global sense.
- **Asynchronous round robin**: Each processor maintains a counter and makes requests in a round-robin fashion.
- **Global round robin**: The system maintains a **global counter** and requests are made in a round-robin fashion, globally.
- **Random polling**: Request a randomly selected processor for work.

Analysis of Load-Balancing Schemes: Conclusions

- **Asynchronous round robin has poor performance** because it makes a large number of work requests.
- Global round robin has poor performance because of **contention at counter**, although it makes the least number of requests.
- Random polling strikes a desirable compromise.

Termination Detection

- **How do you know when everyone's done?**
- A number of algorithms have been proposed.

Dijkstra's Token Termination Detection

- Assume that all processors are **organized in a logical ring**.
- Assume, for now that work **transfers can only happen from P_i to P_j if $j > i$**.
- Processor P_0 **initiates a token** on the ring when it goes idle.
- Each intermediate processor receives this token and **forwards it when it becomes idle**.
- When **the token reaches processor P_0**, all processors are done.

Dijkstra's Token Termination Detection

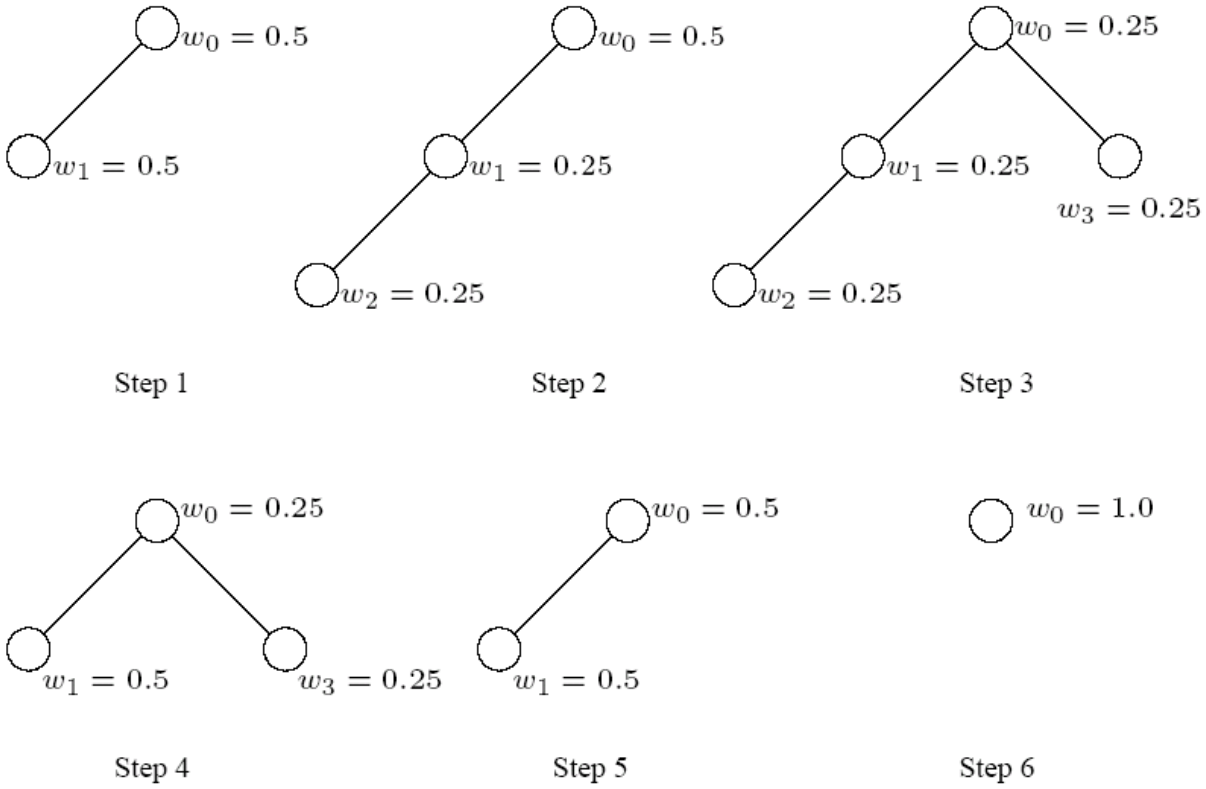
Now, let us do **away with the restriction on work transfers**.

- When processor P_0 goes idle, it **colors itself green** and initiates a green token.
- If processor P_j sends work to processor P_i and $j > i$ then processor P_j **becomes *red***.
- If processor P_i has the token and P_i is idle, it passes the token to P_{i+1}. **If P_i is *red*, then the color of the token is set to *red*** before it is sent to P_{i+1}. **If P_i is *green*, the token is passed unchanged.**
- After P_i passes the token to P_{i+1}, P_i becomes *green* .
- The algorithm **terminates when processor P_0 receives a *green* token** and is itself idle.

Tree-Based Termination Detection

- **Associate weights with individual workpieces**. Initially, processor P_0 has all the work and a weight of **one**.
- Whenever **work is partitioned**, the **weight is split** into half and sent with the work.
- When a **processor gets done** with its work, it **sends its parent the weight back**.
- **Termination** is signaled when the **weight at processor P_0 becomes 1** again.
- Note that underflow and **finite precision** are important factors associated with this scheme.

Tree-Based Termination Detection



Tree-based termination detection. Steps 1-6 illustrate the weights at various processors after each work transfer

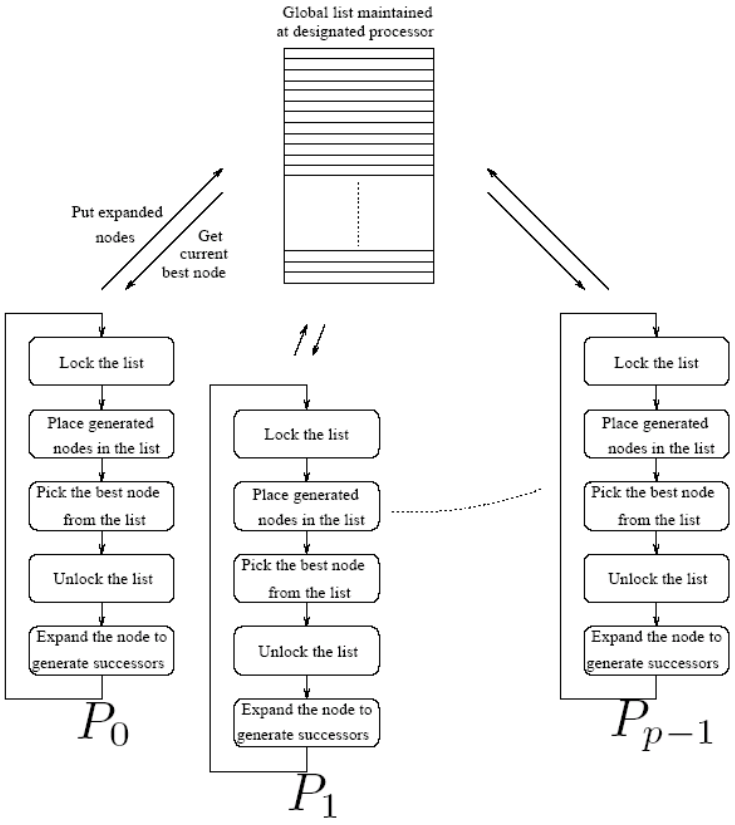
Parallel Formulations of Depth-First Branch-and-Bound

- Parallel formulations of **depth-first branch-and-bound** search (DFBB) are similar to those of DFS.
- Each processor has a **copy of the current best solution**. This is used as a local bound.
- If a processor detects another solution, it **compares the cost with current best solution**. If the cost is better, it broadcasts this cost to all processors.
- If a processor's current best solution path is worse than the globally best solution path, only the efficiency of the search is affected, not its correctness.

Parallel Best-First Search

- The core data structure is the **Open list** (typically implemented as a **priority queue**).
- Each processor **locks this queue**, **extracts the best node**, **unlocks it**.
- **Successors of the node are generated**, their **heuristic functions estimated**, and the nodes **inserted into the open list** as necessary after appropriate locking.
- **Termination signaled** when we find **a solution whose cost is better than the best heuristic value in the open list**.
- Since we expand more than one node at a time, we may expand nodes that would not be expanded by a sequential algorithm.

Parallel Best-First Search

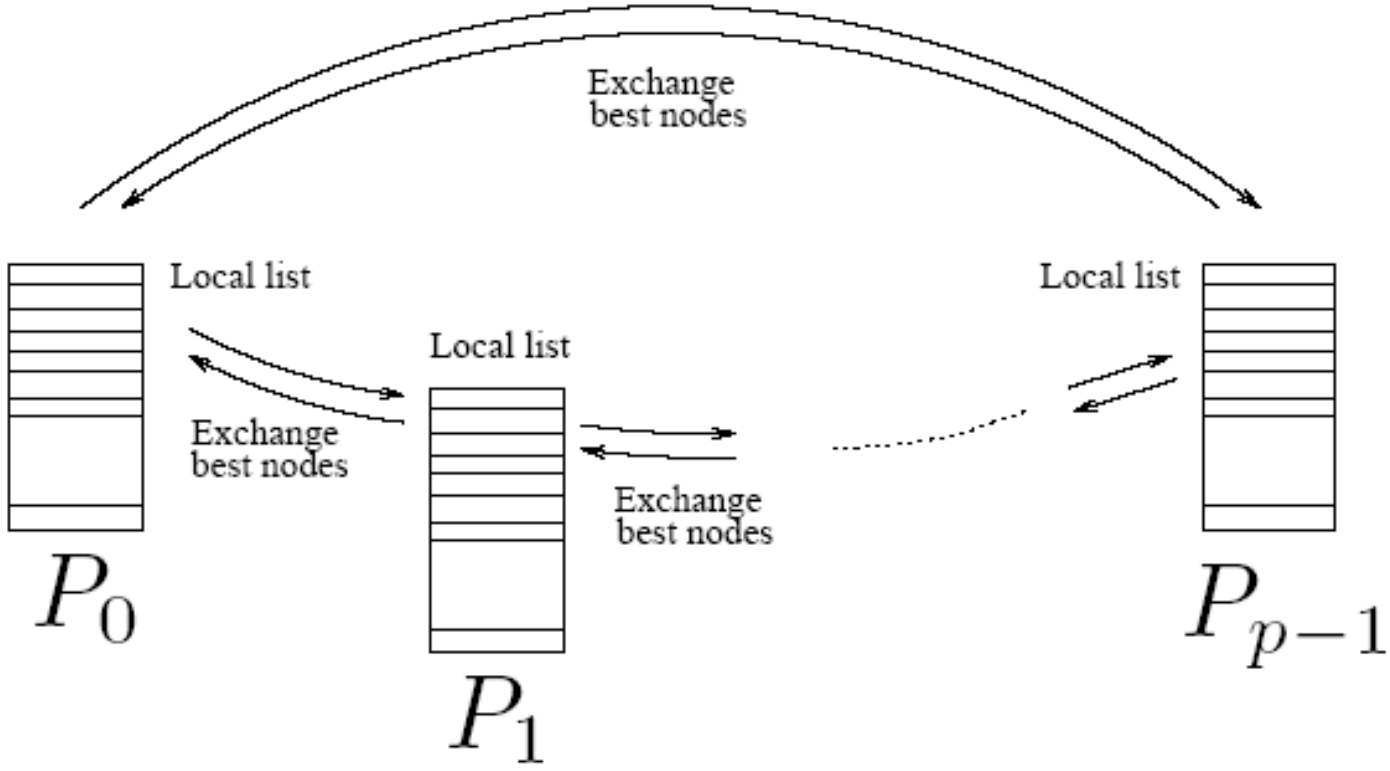


A general schematic for parallel best-first search using a centralized strategy. The **locking operation** is used here to **serialize queue access** by various processors.

Parallel Best-First Search

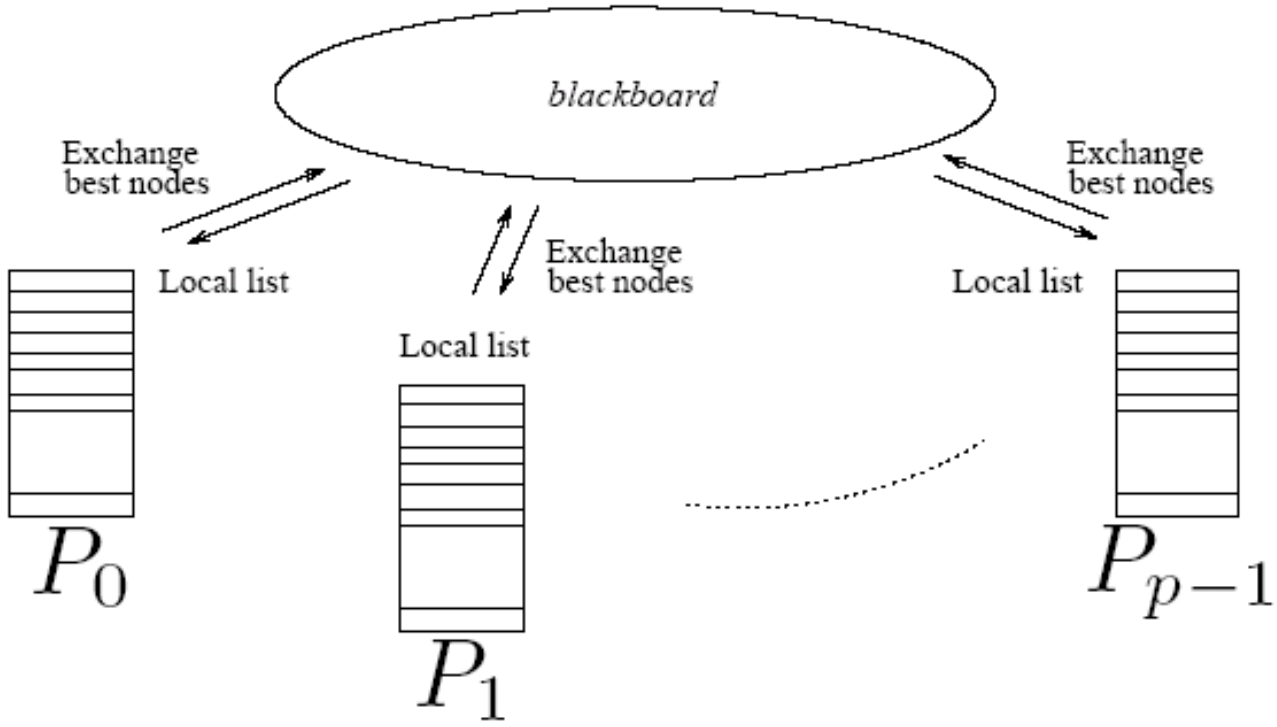
- The **open list is a point of contention**.
- Avoid contention by having **multiple open lists**.
- Initially, the search space is **statically divided** across these open lists.
- **Processors concurrently operate on these open lists**.
- Since the heuristic values of nodes in these lists may diverge significantly, we must periodically **balance the quality of nodes in each list**.
- A number of **balancing strategies** based on **ring**, **blackboard**, or **random communications** are possible.

Parallel Best-First Search



A message-passing implementation of parallel best-first search using the **ring communication strategy**.

Parallel Best-First Search



An implementation of parallel best-first search using the **blackboard** communication strategy.

Parallel Best-First Graph Search

- Graph search involves a **closed list**, where the **major operation is a lookup** (on a key corresponding to the state).
- The classic data structure is a **hash**.
- **Hashing can be parallelized** by using two functions - the first one hashes each node to a processor, and the second one hashes within the processor.
- This strategy can be combined with the idea of multiple open lists.
- **If a node does not exist in a closed list, it is inserted into the open list** at the target of the first hash function.
- In addition to facilitating lookup, randomization also equalizes quality of nodes in various open lists.

Speedup Anomalies in Parallel Search

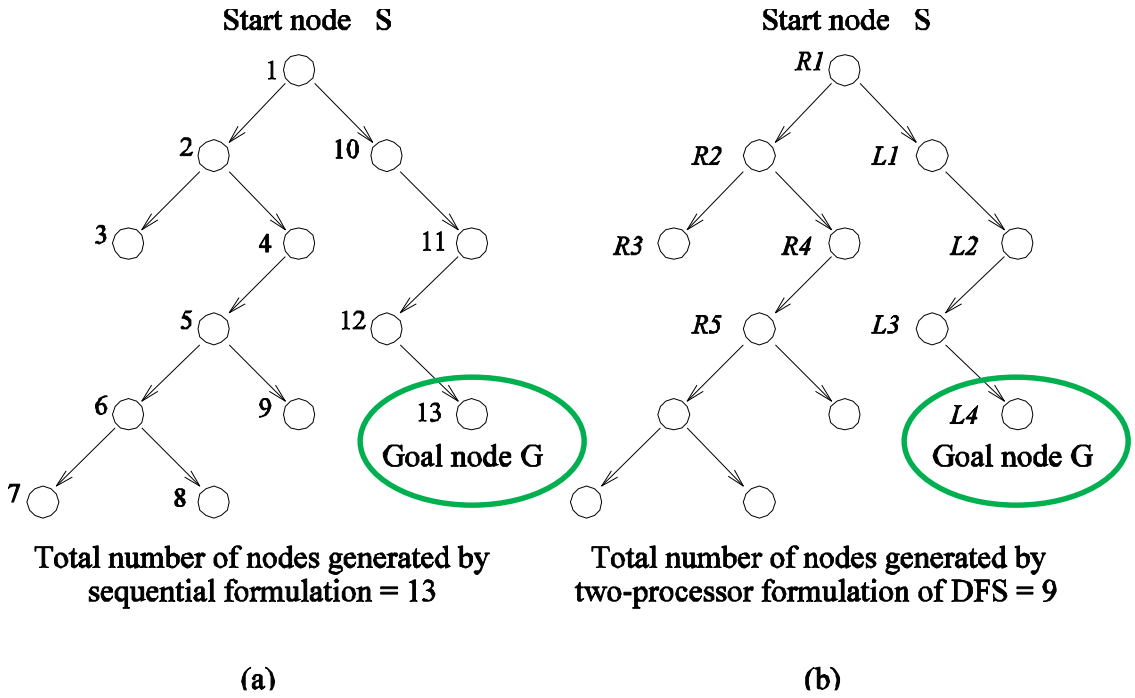
- Since the search space explored by processors is determined dynamically at runtime, the actual **work might vary significantly**.

- Executions yielding speedups greater than p by using p processors are referred to as ***acceleration anomalies***. Speedups of less than p using p processors are called ***deceleration anomalies***.

- Speedup anomalies also manifest themselves in best-first search algorithms.

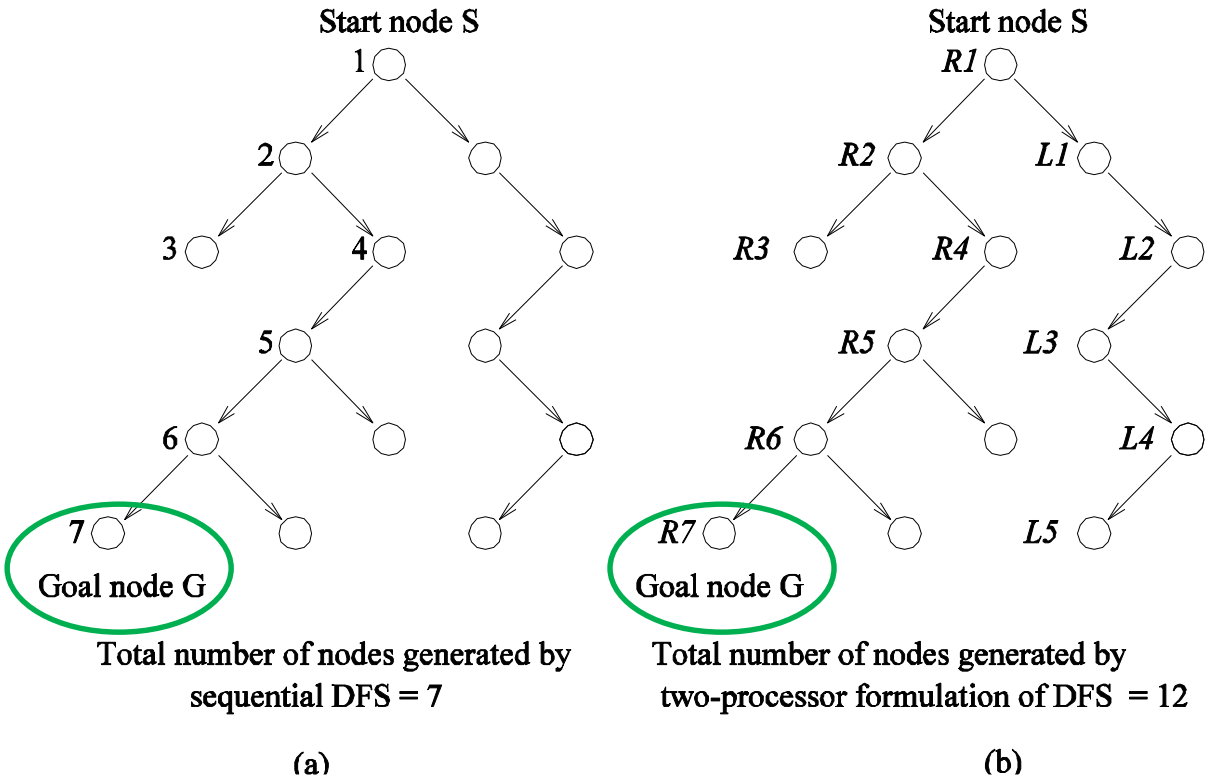
- If the heuristic function is good, the work done in parallel best-first search is typically more than that in its serial counterpart.

Speedup Anomalies in Parallel Search



The difference in number of nodes searched by sequential and parallel formulations of DFS. For this example, **parallel DFS reaches a goal node after searching fewer nodes than sequential DFS**.

Speedup Anomalies in Parallel Search



A parallel **DFS formulation that searches more nodes than its sequential counterpart**.