Logical reasoning and programming, task 2

(November 3, 2025)

Your task is to show some properties of a function using an SMT solver. Although the function is written in Python, you should think about it more like a function from real numbers to real numbers that is defined using Python. The function is

```
def heron(x, init=1, iters=6):
    res = init
    for _ in range(iters):
        res = (x / res + res) / 2
    return res
```

and it implements Heron's method for approximating $\sqrt{x}$ where `init` is the initial estimate and `iters` is the number of iterations.

Problems

1. Let x be a real number such that $0 < x < 100$.
   - Find a counterexample for
     $$\left|\texttt{heron(x, init=1, iters=6)} - \sqrt{x}\right| < 0.01.$$
   - Prove
     $$\left|\texttt{heron(x, init=1, iters=7)} - \sqrt{x}\right| < 0.01.$$

   (5 points.)

2. Let x and b be real numbers such that $0 < x$ and $0 < b$.
   - Find a counterexample for

$$\left|\texttt{heron(x, init=b, iters=1)} - \sqrt{x}\right| > \left|\texttt{heron(x, init=b, iters=2)} - \sqrt{x}\right|.$$

   - Prove that the previous holds if $b \neq \sqrt{x}$,
   - If $b \neq \sqrt{x}$, find (experimentally) the maximal real number c for which you can prove

$$\left|\texttt{heron(x,init=b,iters=1)} - \sqrt{x}\right| > c \cdot \left|\texttt{heron(x,init=b,iters=3)} - \sqrt{x}\right|.$$

   (7 points.)

3. Let x, b_1, and b_2 be real numbers such that $0 < x$ and $0 < b_1 < b_2$. Moreover,
   $$\left|\sqrt{x} - b_1\right| = \left|\sqrt{x} - b_2\right|.$$
   Is `heron(x, iters=1, init=b1)` or `heron(x, iters=1, init=b2)` a better approximation of $\sqrt{x}$? Prove your claim!

   (3 points.)

Software

You may use Z3 or CVC5 to prove claims or find counterexamples. If you want to use a different SMT solver, please, discuss it in advance. The recommended approach is using Z3 API in Python as in the example that you are highly recommended to check. It is possible to install it using

```
pip install z3-solver
```

and the detailed description is available here. Although Z3 API has a `Sqrt(x)` function, using a new real `sqrt_x` (as shown in the example) is preferred. It is also best to avoid quantifiers, as their behavior is somewhat complex.

However, you may use Python API in cvc5 instead.

Solution

You are supposed to submit your solution to BRUTE. Ideally, in a single Python file similar to the provided example. Find a counterexample/prove means using an SMT solver. Note that this task will be evaluated manually.

Please, do submit even incomplete solutions!