Logical reasoning and programming, lab session 6

(October 27, 2025)

6.1 Show that

$$(g(f(x_1 - 2)) = x_1 + 2) \wedge (g(f(x_2)) = x_2 - 2) \wedge (x_2 + 1 = x_1 - 1)$$

is unsatisfiable by the Nelson–Oppen procedure, where x_1 and x_2 are integers and f and g uninterpreted functions.

Why does the procedure work here even though QF_LIA is non-convex?

6.2 If we want to combine theories in SMT using the Nelson–Oppen method, we require that both of them are stably infinite. Assume two theories $\mathcal{T}_1$ with the language $\{f\}$ and $\mathcal{T}_2$ with the language $\{g\}$, where f and g are uninterpreted unary function symbols. Moreover, $\mathcal{T}_1$ has only models of size at most 2 (for example, it contains $\forall X \forall Y \forall Z (X = Y \vee X = Y \vee Y = Z)$ as an axiom). Show that the Nelson–Oppen method says that

$$f(x_1) \neq f(x_2) \wedge g(x_2) \neq g(x_3) \wedge g(x_1) \neq g(x_3).$$

is satisfiable in the union of $\mathcal{T}_1$ and $\mathcal{T}_2$, but this is clearly incorrect.

6.3 Show that the following formulae are valid and provide counter-examples for the opposite implications:

(a) $\forall X p(X) \vee \forall X q(X) \to \forall X (p(X) \vee q(X))$,
(b) $\exists X (p(X) \wedge q(X)) \to \exists X p(X) \wedge \exists X q(X)$,
(c) $\exists X \forall Y p(X, Y) \to \forall Y \exists X p(X, Y)$,
(d) $\forall X p(X) \to \exists X p(X)$.

6.4 Decide whether for any formula φ holds:

(a) $\varphi \equiv \forall \varphi$,
(b) $\varphi \equiv \exists \varphi$,
(c) $\models \varphi$ iff $\models \forall \varphi$,
(d) $\models \varphi$ iff $\models \exists \varphi$,

where $\forall \varphi$ ($\exists \varphi$) is the universal (existential) closure of φ. If not, does at least one implication hold?

6.5 Show that for any set of formulae Γ and a formula φ holds if $\Gamma \models \varphi$, then $\forall \Gamma \models \varphi$, where $\forall \Gamma = \{ \forall \psi \colon \psi \in \Gamma \}$. Does the opposite direction hold?

6.6 Does it hold $\Gamma \models \varphi$ iff $\forall \Gamma \models \forall \varphi$?

6.7 Find a set of formulae Γ and a formula φ such that $\Gamma \models \varphi$ and $\Gamma \models \neg \varphi$.

6.8 Produce equivalent formulae in prenex form:

(a) $\forall X (p(X) \to \forall Y (q(X, Y) \to \neg \forall Z r(Y, Z)))$,
(b) $\exists X p(X, Y) \to (q(X) \to \neg \forall Z p(X, Z))$,

(c) $\exists X p(X,Y) \rightarrow (q(X) \rightarrow \neg\exists Z p(X,Z))$,

(d) $p(X,Y) \rightarrow \exists Y(q(Y) \rightarrow (\exists X q(X) \rightarrow r(Y)))$,

(e) $\forall Y p(Y) \rightarrow (\forall X q(X) \rightarrow r(Z))$.

6.9 In **6.8** you could obtain in some cases various prefixes; the order of quantifiers can be different. Are all these variants correct?

6.10 Can we produce a formula equivalent to **6.8**e with just one quantifier?

6.11 Produce Skolemized formulae equisatisfiable with those in **6.8**. Try to produce as simple as possible Skolem functions.

6.12 Skolemize the following formula

$$\forall X(p(a) \vee \exists Y(q(Y) \wedge \forall Z(p(Y,Z) \vee \exists U \neg q(X,Y,U)))) \vee \exists W q(a,W).$$

Why is it possible in this particular case to do that without producing an equivalent formula in prenex form?

6.13 We say that a binary predicate q is the transitive closure of a binary predicate p, if $q(s,t)$ iff there is a sequence of terms $s = t_1, t_2, \ldots, t_{n-1}, t_n = t$ such that $p(t_i, t_{i+1})$, for $1 \leq i < n$. Is the formula

$$\forall X \forall Z(q(X,Z) \leftrightarrow (p(X,Z) \vee \exists Y(p(X,Y) \wedge q(Y,Z))))$$

a correct definition of q?

6.14 The compactness theorem in First-Order Logic says that a set of sentences has a model iff every finite subset of it has a model. Use this theorem to show that the transitive closure is not definable in FOL.

Hint: Assume for a contradiction that φ is a formula that expresses that q is the transitive closure of p. Let $\psi^n(a,b) = \neg(\exists X_1 \ldots \exists X_{n-1}(p(a,X_1) \wedge p(X_1,X_2) \wedge \cdots \wedge p(X_{n-1},b))$ (Hence $\psi^1(a,b)$ means $\neg p(a,b)$ and $\psi^2(a,b)$ means $\neg(\exists X_1(p(a,X_1) \wedge p(X_1,b))))$. What can you say about the satisfiability of $\Gamma = \{\varphi\} \cup \{q(a,b)\} \cup \{\psi^1(a,b), \psi^2(a,b), \ldots\}$?