Logical reasoning and programming, lab session 1

(September 22, 2025)

1.1 Decide which of the following formulae are tautologies:

(a) $((p \to q) \to q) \to q$,
(b) $((p \to q) \to p) \to p$,
(c) $(p \to q) \vee (q \to p)$,
(d) $((p \to q) \wedge q) \to p$,
(e) $\neg p \to \neg(p \vee (p \wedge q))$.

Although formulae contain only two variables, try to find a better method than truth tables if possible.

1.2 Decide which of the following claims are true:

(a) $\neg(\varphi \leftrightarrow \psi) \models \varphi \wedge \psi$,
(b) $\neg(\varphi \leftrightarrow \psi) \models \varphi \vee \psi$,
(c) $\neg(\varphi \leftrightarrow \psi) \models \varphi \to \psi$.

1.3 If $\varphi \to \psi \in \texttt{TAUT}$ and $\psi \to \chi \in \texttt{TAUT}$, then $\varphi \to \chi \in \texttt{TAUT}$. Why? Does the claim still hold if we replace $\texttt{TAUT}$ with $\texttt{SAT}$ and why?

1.4 Show that $\models \varphi \leftrightarrow \psi$ iff $\varphi \equiv \psi$.

1.5 Are the following two C programs equivalent?

```
if(!a && !b) h();
else if(!a) g();
else f();
```

```
if(a) f();
else if(b) g();
else h();
```

Prove their equivalence formally, or provide a counterexample. Please, check your solution against these slides.

1.6 Let $\mathrm{Cl}(\Gamma) = \{\, \varphi \colon \Gamma \models \varphi \,\}$. What is $\mathrm{Cl}(\emptyset)$ equivalent to? Let Γ and Δ be sets of formulae. Check whether:

(a) $\Gamma \subseteq \mathrm{Cl}(\Gamma)$,
(b) $\mathrm{Cl}(\mathrm{Cl}(\Gamma)) = \mathrm{Cl}(\Gamma)$,
(c) $\mathrm{Cl}(\Gamma \cup \Delta) = \mathrm{Cl}(\Gamma) \cup \mathrm{Cl}(\Delta)$.

If the equality does not hold in (b) or (c), does at least one of the inclusions hold?

1.7 Recall that a Boolean function of n-variables is a function $f\colon \{0,1\}^n \to \{0,1\}$. Describe all functions of one variable. How many distinct Boolean functions of n variables exist? Try $n = 0, 1, 2, \ldots$ first. Do you know, why are functions NAND ($\uparrow$) and NOR ($\downarrow$) interesting?[1]

[1] These connectives are also called Sheffer stroke and Peirce arrow, respectively. They are defined as $x \uparrow y := \neg(x \wedge y)$ and $x \downarrow y := \neg(x \vee y)$.

1.8 If we use standard rewriting rules for producing a CNF, then we usually conclude by some simplifications—remove duplicate clauses and literals. Why can we do that? Is it correct that there is no need for a variable to occur more than once in a clause?

1.9 Produce a formula in CNF which is equivalent to

$$\varphi = (a \to (c \land d)) \lor (b \to (c \land e)).$$

Then use the Tseytin transformation to produce a formula in CNF which is equisatisfiable to φ.

1.10 Use Boolean formula manipulation in PySAT or Limboole on the previous formula $\varphi = (a \to (c \land d)) \lor (b \to (c \land e))$.

1.11 Why is the formula produced by Tseytin transformation only equisatisfiable and not equivalent to the original formula?

1.12 Try solving SAT problems by hand in The SAT Game.