Digital Image

(B4M33DZO)

Lecture 3:

Convolution

https://cw.fel.cvut.cz/wiki/courses/dzo/start

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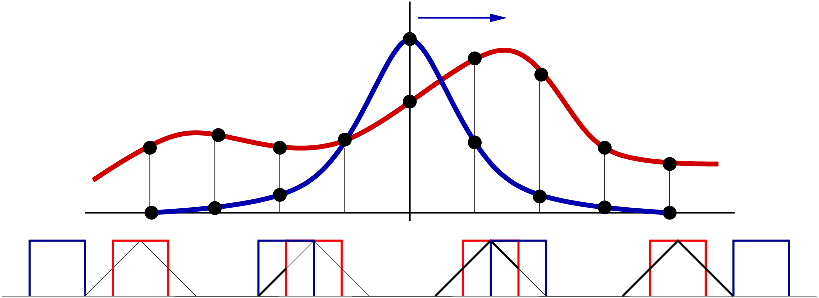
Convolution

Sliding average of function f weighted by function g:

Definition

$$(f * g)(t) = \int_{-\infty}^{\infty} f(x) \cdot g(t - x)\,\mathrm{d}x$$



Interesting Properties

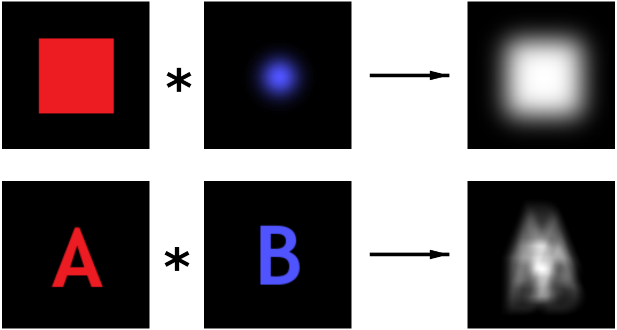
- **Commutativity:**
$$f * g = g * f$$
- **Associativity:**
$$f * (g * h) = (f * g) * h$$
- **Distributivity:**
$$f * (g + h) = (f * g) + (f * h)$$
- **Differentiation:**
$$(f * g)' = f' * g = f * g'$$
- **Convolution theorem:**
$$f * g = \mathcal{F}^{-1}\{\mathcal{F}\{f\} \cdot \mathcal{F}\{g\}\}$$

Convolution (in 2D) — Continuous Case

Extension of 1D case (image F, convolution kernel G):

Definition

$$(F * G)(s,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y) \cdot G(s-x, t-y)\,\mathrm{d}x\,\mathrm{d}y$$



Convolution (in 2D) — Discrete Case

Discrete convolution (for image F and kernel G):

$$(F * G)[s,t] = \sum_x \sum_y F[x][y] \cdot G[s-x][t-y]$$

Computational complexity: $\mathcal{O}(|F| \cdot |G|)$

Would it be possible to compute it faster?

Methods:

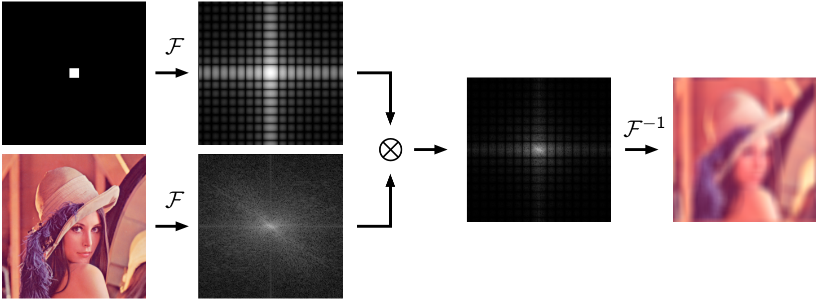
Method	**Accuracy**	**Kernel**	**Complexity**
Fourier Transform	Exact	Arbitrary	$\mathcal{O}(\lvert F\rvert \cdot \log \lvert F\rvert)$
Separable Kernel	Exact	Limited	$\mathcal{O}(\lvert F\rvert \cdot \sqrt{\lvert G\rvert})$
Integral Image	High	Limited	$\mathcal{O}(\lvert F\rvert)$

Fourier Transform

Convolution Theorem:

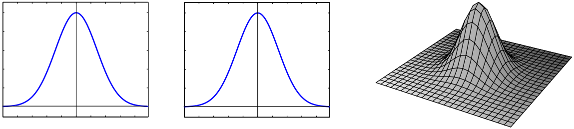
Theorem

$$\mathcal{F}\{F * G\} = \mathcal{F}\{F\} \cdot \mathcal{F}\{G\}$$



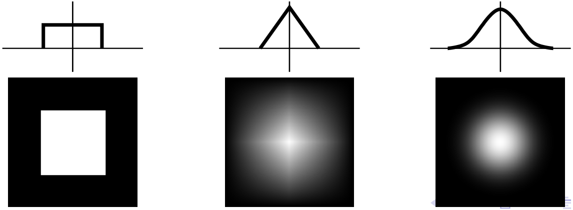
Separable Kernel

$$G(x, y) = g_x(x) \cdot g_y(y)$$



Separable Convolution

$$(F * G)(s, t) = \int_{-\infty}^{\infty} g_y(t - y) \left(\int_{-\infty}^{\infty} F(x, y) \cdot g_x(s - x)\, \mathrm{d}x \right) \mathrm{d}y$$



Integral Image

Summed Area Table (SAT):

$$S[x,y] = \sum_{x' \leq x} \sum_{y' \leq y} I[x', y']$$

2	3	2	1
3	0	1	2
1	3	1	0
1	4	2	2

Image I

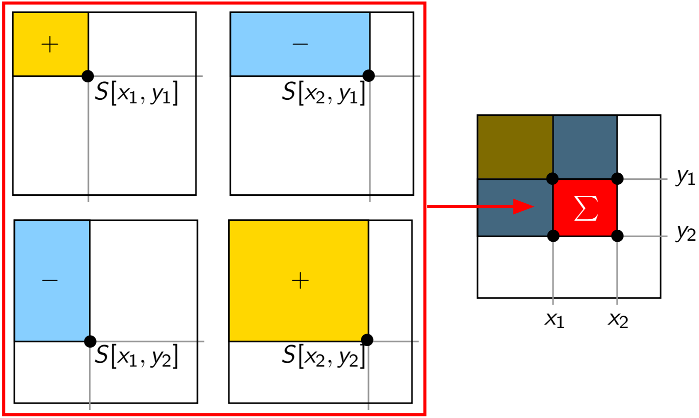
2	5	7	8
5	8	11	14
6	12	16	19
7	17	23	28

SAT S

Integral Image

Region Sum:

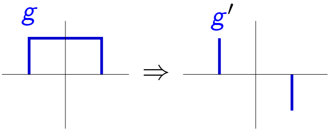
$$\Sigma = S[x_2, y_2] - S[x_1, y_2] - S[x_2, y_1] + S[x_1, y_1]$$



Integral Image — Extension

Basic Idea

$$f * g = [\int f] * [g']$$



Extension for more complex kernels:

Generalized Form

$$f * g = [\int^{(n)} f] * [g^{(n)}]$$

Equivalently,

$$(\underbrace{g * g * \cdots * g}_{n})^{(n)} = \underbrace{g' * g' * \cdots * g'}_{n}$$

Integral Image — Extension

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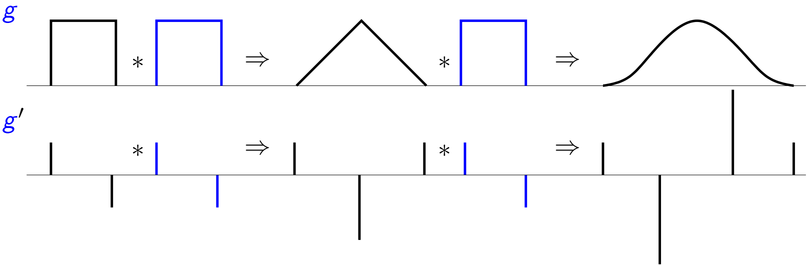


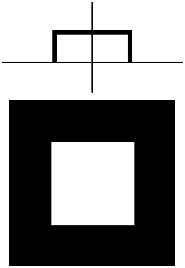
Integral Image — Kernel Examples

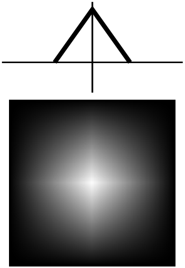
Examples of convolution kernels:

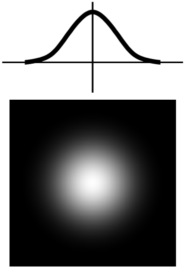
-1	1
1	-1

1	-2	1
-2	4	-2
1	-2	1

-1	3	-3	1
3	-9	9	-3
-3	9	-9	3
1	-3	3	-1







Fourier Transform — Pros and Cons

Advantages:

- Arbitrary kernel shape
- Accurate convolution
- Suitable for large kernels
- Independent of kernel width

Disadvantages:

- Two FFTs per convolution
- Slow for small kernels

Separable Kernel — Pros and Cons

Advantages:

- Accurate convolution
- Speed-up for large kernels

Disadvantages:

- Limited kernel shape
- Still slow for very large kernels
- Requires extra memory and two passes

Integral Image — Pros and Cons

Advantages:

- Fast
- High accuracy

Disadvantages:

- Limited numeric precision (when approximation is used)
- Limited kernel shapes