$\boxed{\text{Equivalent conditions for Nash equilibrium in matrix games}}$

1) $U(p_1, p_2^*) \le U(p_1^*, p_2^*) \le U(p_1^*, p_2)$

$\forall p_1 \in \Delta_1, \forall p_2 \in \Delta_2$

2) $U(s_1, p_2^*) \le U(p_1^*, p_2^*) \le U(p_1^*, s_2)$

$\forall s_1 \in S_1 \; \forall s_2 \in S_2$

3) $\max\limits_{s_1 \in S_1} U(s_1, p_2^*) = U(p_1^*, p_2^*) = \min\limits_{s_2 \in S_2} U(p_1^*, s_2)$

4) $\max\limits_{p_1 \in \Delta_1} \min\limits_{s_2 \in S_2} U(p_1, s_2) \quad =: \underline{v}$

$=$

$U(p_1^*, p_2^*) \quad =: v$

$=$

$\min\limits_{p_2 \in \Delta_2} \max\limits_{s_1 \in S_1} U(s_1, p_2^*) \quad =: \overline{v}$

Double Oracle – correctness

1) $v \le v_u := U(s_1, g_2^*)$ since

$$\min_{p_2 \in \Delta_2} \max_{s_1' \in S_1} U(s_1', p_2) = v$$

$$\mid\wedge$$

$$\max_{s_1' \in S_1} U(s_1', g_2^*) = v_u$$

2) Similarly

3) Let $s_i \in T_i$, $i = 1, 2$.

Then

NE of the subgame

$$U(g_1^*, g_2^*) = \max_{s_1' \in T_1} U(s_1', g_2^*)$$

Assumption $s_1 \in T_1$

$$= \max_{s_1' \in S_1} U(s_1', g_2^*) = v_u$$

and $U(g_1^*, g_2^*) = v_e$ analogously.

This implies that $v = U(g_1^*, g_2^*)$ and (g_1^*, g_2^*) is a NE.

Alternative terminating condition

Let $v_u - v_\ell < \varepsilon$ for some $\varepsilon > 0$. We will show that (g_1^*, g_2^*) is an ε-NE:

$$U(g_1^*, g_2^*) \ge \overbrace{U(g_1^*, s_2)}^{v_\ell} \overset{\text{Assumption}}{\ge} \underbrace{U(s_1, g_2^*)}_{v_u} - \varepsilon = \max_{s_1' \in S_1} U(s_1', g_2^*) - \varepsilon$$

$$\ge U(s_1, g_2^*) - \varepsilon \quad \forall s_1 \in S_1$$

The second inequality follows analogously.