NORMAL-FORM GAMES

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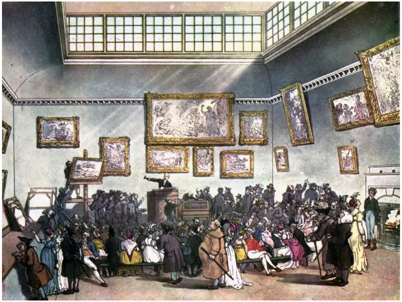
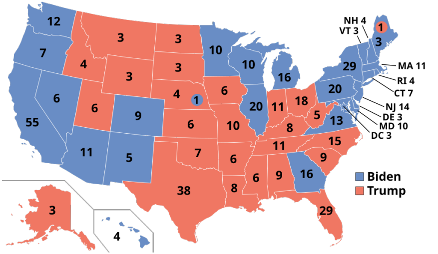
GAME THEORY

Mathematical theory of interactive decision-making

Players ⟶ Strategies ⟶ Outcomes ⟶ Utilities





Wikipedia

STORIES OF SUCCESS

Foundations

- von Neumann (1928): Minimax theorem - birth of modern game theory
- Nash (1951): Nash equilibrium - universal solution concept
- Shapley (1953): Fair division of cooperative gains

Economic/social impact

- Vickrey (1961): Second-price auctions - foundation of ad markets
- Myerson (1981): Mechanism design - Nobel-winning contribution

AI milestones in games

- DeepStack (2017): Expert-level AI in poker
- AlphaGo (2016) / AlphaStar (2019): Deep RL triumphs

PLAYERS - FROM HUMANS TO AGENTS

Strategic interaction among humans (bargaining, markets) is increasingly giving way to interaction among agents (algorithms, bots, platforms):

- Stock markets run by trading algorithms
- Uber/airline prices set dynamically
- Recommender systems influence choices
- LLMs are vulnerable to strategic manipulation
- Neural nets compete to produce better outputs

GAME THEORY AS AN EXTENSION OF OPTIMIZATION

Optimization

- One decision-maker
- Goal: maximize/minimize an objective
- Assumes full control over variables
- *What is my best decision?*

Game Theory

- Multiple decision-makers
- Each optimizes their own objective
- Outcomes depend on others' choices
- *What is my best decision given that others are optimizing too?*

CLASSIFICATION OF GAMES

Game forms

- normal
- extensive
- coalitional

Dynamics

- static
- dynamic

Information during play

- perfect
- imperfect

Number of strategies

- finite
- infinite

Utility functions

- zero-sum
- general-sum

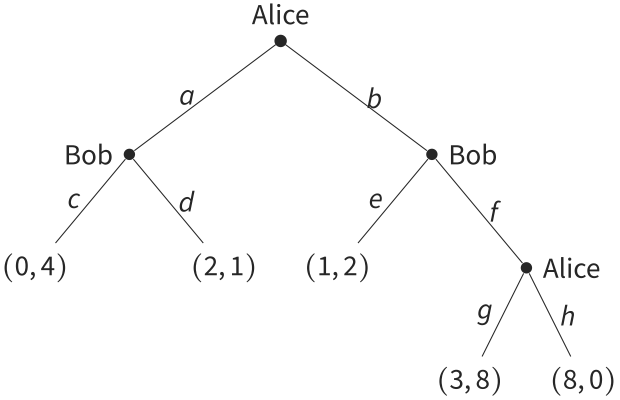
Information about games's structure

- complete
- incomplete

Competitive/cooperative

A GAME WITH PERFECT INFORMATION

Alice meets Bob for breakfast/lunch. Bob chooses for breakfast Starbucks or McDonald's, and a lunch venue. They might go to a Czech restaurant or a pizzeria, where Alice chooses between Sicilian or Neapolitan pizza.



Which venue will be chosen?

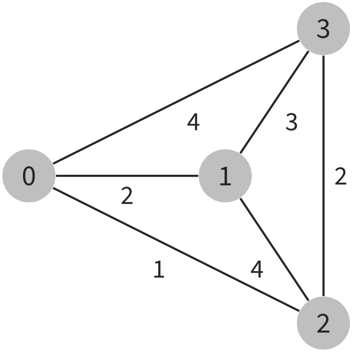
FIRST-PRICE SEALED-BID AUCTION

- A single item is up for auction
- Each buyer submits a bid simultaneously in a sealed envelope
- The item is given to the highest bidder who pays her bid

What is the optimal bidding strategy for each buyer?

COOPERATIVE GAME

- Cities 1,2, and 3 need to connect to the provider of energy 0
- The graph shows costs of pairwise connections



How to allocate the costs?

OUTLINE OF THE COURSE

Topic	*Lectures*	*Motivation*
normal-form games	5	fundamentals
extensive-form games	2	card and board games
games with incomplete information	3	auctions
cooperative games	3	cost allocation, voting

NORMAL-FORM GAMES

NORMAL-FORM GAMES

Definition

Normal-form (or strategic) game is a triple $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$, where

- $N = \{1, \ldots, n\}$ is a player set
- S_i is a strategy set, for each player $i \in N$
- u_i is a utility function

$$u_i \colon \mathbf{S} \to \mathbb{R}$$

of player $i \in N$, where $\mathbf{S} = S_1 \times \cdots \times S_n$

HOW A NORMAL-FORM GAME IS PLAYED

- Each player independently chooses one of their available strategies s_i, without knowing the choices of others.
- Once all players have chosen, a strategy profile $(s_1, \dots, s_n)$ is revealed.
- The resulting utility of player i is $u_i(s_1, \dots, s_n)$.

EXAMPLE

Prisoner's dilemma

The police make the following offer to two prisoners. If one squeals on the other that both committed the serious crime, then the confessor will be set free and the other will spend 4 years in jail. If both confess, then they will each get the 3-year sentence. If both stay quiet, then they will each spend 1 year in jail for the minor offense.

	q	s
q	$-1, -1$	$-4, 0$
s	$0, -4$	$-3, -3$

TWO-PLAYER ZERO-SUM GAMES

- $N = \{1, 2\}$
- $u_1 + u_2 = 0$

The value of $u_1(s_1, s_2)$ is being tugged away at from two sides, by player 1 who wants to maximize it, and by player 2 who wants to minimize it.

von Neumann (1928)

Matching pennies

Each player places a penny on a table, heads up or tails up. If the pennies match, player 1 wins; otherwise player 2 wins.

	h	t
h	1	−1
t	−1	1

FIRST-PRICE SEALED-BID AUCTION

One object is for sale. Each buyer i submits a bid $b_i \geq 0$ simultaneously in a sealed envelope. The item is given to the *highest* bidder who pays her bid.

- Every bidder attaches a private value $v_i > 0$ to the item.
- The winner is selected uniformly at random from the k highest bidders.

Strategic game representation

Strategy sets $S_i = [0, \infty)$ and utility functions

$$u_i(b_1, \ldots, b_n) = \begin{cases} 0 & b_i < \overline{b}_i, \\ (v_i - \overline{b}_i)/k & b_i = \overline{b}_i, \\ v_i - b_i & b_i > \overline{b}_i, \end{cases}$$

where $\overline{b}_i := \max_{j \neq i} b_j$.

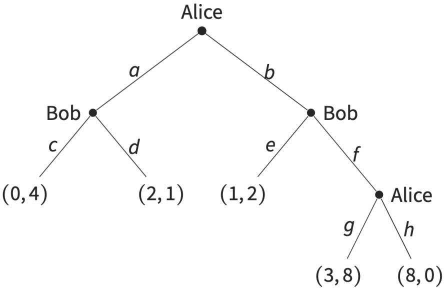
SECOND-PRICE SEALED-BID AUCTION

The rules are the same as in the first-price auction, except that the winner pays the price equal to the *second-highest* bid.

Strategic game representation

$$u_i(b_1, \ldots, b_n) = \begin{cases} 0 & b_i < \overline{b}_i, \\ (v_i - \overline{b}_i)/k & b_i = \overline{b}_i, \\ v_i - \overline{b}_i & b_i > \overline{b}_i. \end{cases}$$

FROM EXTENSIVE-FORM GAMES TO STRATEGIC GAMES



Bob

Alice		ce	cf	de	df
	ag	0, 4	0, 4	2, 1	2, 1
	ah	0, 4	0, 4	2, 1	2, 1
	bg	1, 2	3, 8	1, 2	3, 8
	bh	1, 2	8, 0	1, 2	8, 0

SOLUTION CONCEPTS FOR STRATEGIC GAMES

RATIONALITY

A person's behavior is rational if it is in his best interests, given his information. R. Aumann (2006)

- The utilitaristic concept of rationality imposes no restrictions on the norms of human behavior

Both parties deprecated war; but one would make war rather than let the nation survive; and the other would accept war rather than let it perish. And the war came. A. Lincoln (1865)

- We assume that rational players maximize utility

DOMINATION OF STRATEGIES

Which strategies a player never uses?

Definition

Let $s, t \in S_i$ be strategies of player i. We say that

- t is strictly dominated by s if $u_i(s, \mathbf{s}_{-i}) > u_i(t, \mathbf{s}_{-i})$ for every $\mathbf{s}_{-i} \in \mathbf{S}_{-i}$,
- t is weakly dominated by s if $u_i(s, \mathbf{s}_{-i}) \geq u_i(t, \mathbf{s}_{-i})$ for any $\mathbf{s}_{-i} \in \mathbf{S}_{-i}$ and $u_i(s, \mathbf{s}_{-i}) > u_i(t, \mathbf{s}_{-i})$ for some $\mathbf{s}_{-i} \in \mathbf{S}_{-i}$.

A rational player never uses a strictly dominated strategy.

ITERATED ELIMINATION OF DOMINATED STRATEGIES

- $\ell := 0$, $S_i^0 := S_i$ for each $i \in N$
- Repeat
  1. For each $i \in N$, pick any *strictly* dominated strategy $d_i^\ell \in S_i^\ell$ in the *subgame* given by $S_1^\ell, \ldots, S_n^\ell$
  2. $S_i^{\ell+1} := S_i^\ell \setminus \{d_i^\ell\}$ for each $i \in N$
  3. $\ell := \ell + 1$
- Until no strictly dominated strategy is found for any player

EXAMPLE

$$
\begin{array}{cc|c|c|c|}
 & & & \text{Bob} & \\
 & & c & d & e \\
\hline
\text{Alice} & a & 1,0 & 1,2 & 0,1 \\
 & b & 0,3 & 0,1 & 2,0 \\
\end{array}
$$

1. Bob is rational.

$$
\begin{array}{c|c|c|}
 & c & d \\
a & 1,0 & 1,2 \\
b & 0,3 & 0,1 \\
\end{array}
$$

2. Alice is rational and she knows 1.

$$
\begin{array}{c|c|c|}
 & c & d \\
a & 1,0 & 1,2 \\
\end{array}
$$

3. Bob is rational and he knows 2.

$$
\begin{array}{c|c|}
 & d \\
a & 1,2 \\
\end{array}
$$

EXAMPLES

Matching pennies

	h	t
h	1	-1
t	-1	1

No strategy is dominated.

Bach or Stravinski

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

No strategy is dominated.

Prisoner's dilemma

	q	s
q	$-1, -1$	$-4, 0$
s	$0, -4$	$-3, -3$

After the elimination: only s

Coordination game

	a	b
a	$2, 2$	$0, 0$
b	$0, 0$	$1, 1$

No strategy is dominated.

PROPERTIES OF ITERATED ELIMINATION

- The resulting sets S_i^{ℓ} do not depend on the *order* of elimination
- However, the order of elimination is crucial when the algorithm is modified to remove *weakly* dominated strategies!

Some games implicitly include a weakly dominant strategy for each player.

TRUTHFUL BIDDING IN SECOND-PRICE AUCTIONS

Strategic game representation

Strategy sets $S_i = [0, \infty)$ and utility functions

$$u_i(b_1, \ldots, b_n) = \begin{cases} 0 & b_i < \overline{b}_i, \\ (v_i - \overline{b}_i)/k & b_i = \overline{b}_i, \\ v_i - \overline{b}_i & b_i > \overline{b}_i. \end{cases}$$

Proposition

The strategy v_i is weakly dominant in the second-price auction.

NASH EQUILIBRIUM

A strategy profile in which no player has an incentive to deviate, assuming the strategies of all other players remain unchanged.

Definition

A pure strategy profile $\mathbf{s}^* \in \mathbf{S}$ is a Nash equilibrium if for every player $i \in N$

$$u_i(\mathbf{s}^*) \geq u_i(s, \mathbf{s}^*_{-i}) \qquad \text{for all } s \in S_i.$$

NASH EQUILIBRIUM – EXAMPLES

Matching pennies

	h	t
h	1	-1
t	-1	1

There is no NE in pure strategies.

Bach or Stravinski

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

(B, B), (S, S) are the only pure NE.

Prisoner's dilemma

	q	s
q	$-1, -1$	$-4, 0$
s	$0, -4$	$-3, -3$

(s, s) is the only NE.

Coordination game

	a	b
a	$2, 2$	$0, 0$
b	$0, 0$	$1, 1$

(a, a),(b, b) are the only pure NE.

NASH EQUILIBRIUM, EQUIVALENTLY

Which strategy will player i adopt as a reply to the opponents' strategies?

- Best response mapping

$$\mathrm{BR}_i(\mathbf{s}_{-i}) = \operatorname*{argmax}_{s \in S_i} u_i(s, \mathbf{s}_{-i}) \qquad \text{for all } \mathbf{s}_{-i} \in \mathbf{S}_{-i}$$

- Best response to $\mathbf{s}_{-i}$ is any strategy $s \in \mathrm{BR}_i(\mathbf{s}_{-i})$

Proposition

The following are equivalent for a strategy profile $\mathbf{s}^* \in \mathbf{S}^*$.

1. $\mathbf{s}^*$ is a Nash equilibrium.
2. $s_i^* \in \mathrm{BR}_i(\mathbf{s}_{-i}^*)$, for each $i \in N$.

PARETO OPTIMALITY

Which strategy profiles are socially efficient?

Definition

A strategy profile $\mathbf{t} \in \mathbf{S}$ is Pareto optimal if there is no $\mathbf{s} \in \mathbf{S}$ such that $u_i(\mathbf{s}) \geq u_i(\mathbf{t})$ for all $i \in N$ and there is some $i \in N$ such that $u_i(\mathbf{s}) > u_i(\mathbf{t})$.

- No player's utility can be increased without simultaneously decreasing the utility to another player
- Individual preferences of players are ignored

HOW TO FIND A PARETO OPTIMAL PROFILE?

Maximizing social welfare

Define $w \colon \mathbf{S} \to \mathbb{R}$ as

$$w(\mathbf{s}) = \sum_{i \in N} u_i(\mathbf{s}), \quad \mathbf{s} \in \mathbf{S}.$$

Every maximizer of w is Pareto optimal.

- Pareto optimal strategy profile exists in every *finite* strategic game
- All strategy profiles in a zero-sum game are Pareto optimal

PARETO OPTIMALITY – EXAMPLES

Matching pennies

	h	t
h	1	-1
t	-1	1

Every profile is Pareto optimal.

Bach or Stravinski

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

(B, B), (S, S) are Pareto optimal.

Prisoner's dilemma

	q	s
q	$-1, -1$	$-4, 0$
s	$0, -4$	$-3, -3$

Only (s, s) is *not* Pareto optimal.

Coordination game

	a	b
a	$2, 2$	$0, 0$
b	$0, 0$	$1, 1$

(a, a) is Pareto optimal.

CONCLUSIONS

- Typical games have few or no dominated strategies
- A game may have multiple NE with different utility outcomes
- Pareto optimal solution may not be attainable
- Matching Pennies has no definitive solution in pure strategies, requiring a *more general strategy concept*