NASH EQUILIBRIA

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2×2 STRATEGIC GAMES - NASH EQUILIBRIA

	L	R
U	a, b	c, d
D	e, f	g, h

Proposition

1. *If $a \geq e$ and $b \geq d$, then (U, L) is a NE.*
2. *If $e \geq a$ and $f \geq h$, then (D, L) is a NE.*
3. *If $c \geq g$ and $d \geq b$, then (U, R) is a NE.*
4. *If $g \geq c$ and $h \geq f$, then (D, R) is a NE.*

	Listen	*Sleep*
Prepare	$10^6, 10^6$	$-10, 0$
Slack off	$0, -10$	$0, 0$

Professor's Dilemma (A. Procaccia)

The prof chooses a row strategy and the students choose a column strategy.

Which of the two equilibria will arise?

NASH EQUILIBRIA IN AUCTIONS

Second-price auction

- Thruthful bidding v_i is a weakly dominant strategy for each player i, where v_i is a private value of player i.
- This shows that $(v_1, \ldots, v_n)$ is a Nash equilibrium, though not the unique one.

First-price auction

There is no pure NE even if everyone knows everyone's private value.

PURE STRATEGIES ARE INSUFFICIENT

Matching pennies

	h	t
h	1	−1
t	−1	1

Penalty kicks

- Kicker and Goalkeeper
- unnatural or natural side
- Ratio of scored penalties

	u	n
u	0.58	0.95
n	0.93	0.70

Good Samaritan Game

- A cry for aid echoes from a nearby dark alley
- Providing aid takes effort and your utility is 9
- You get 10 utils if someone else helps
- If nobody helps, utility is 0

$$u_i(\mathbf{s}) = \begin{cases} 9 & s_i = a \\ 10 & s_i = \bar{a},\ s_j = a \quad \exists j \neq i \\ 0 & \mathbf{s} = (\bar{a}, \ldots, \bar{a}) \end{cases}$$

MIXED STRATEGIES AND EXPECTED UTILITY

Assumption: Finite strategy set S_i for each player $i \in N$

Definition

Mixed strategy of player i is a probability distribution p_i on S_i.

Expected utility of player i is

$$U_i(p_1, \dots, p_n) = \sum_{\mathbf{s} \in \mathbf{S}} u_i(\mathbf{s}) \prod_{i \in N} p_i(s_i).$$

EXPECTED UTILITY IN A TWO-PLAYER GAME

Example

		Player 2		
		x	y	z
Player 1	a	1,7	1,5	3,4
	b	2,3	0,4	0,6

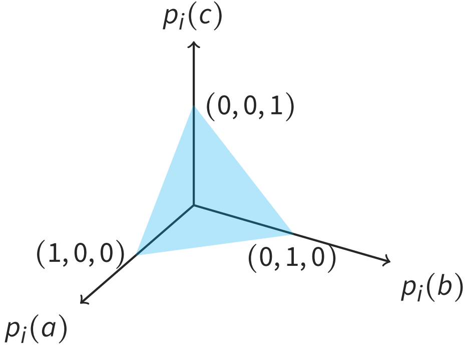
$p \coloneqq p_1(a)$

$q \coloneqq p_2(x), r \coloneqq p_2(y)$

$$\begin{aligned} U_1(p,q,r) &= pq + pr + 3p(1-q-r) + 2(1-p)q \\ U_2(p,q,r) &= 7pq + 5pr + 4p(1-q-r) \\ &\quad + 3(1-p)q + 4(1-p)r + 6(1-p)(1-q-r) \end{aligned}$$

THE SET OF MIXED STRATEGIES

$$\Delta_i \coloneqq \left\{p\colon S_i \to [0,1] \mid \sum_{s\in S_i} p_i(s) = 1\right\}$$



$S_i = \{a, b, c\}$

2-dimensional standard simplex

The three vertices represent pure strategies

NASH EQUILIBRIUM IN MIXED STRATEGIES

A mixed strategy profile

$$\mathbf{p}^* = (p_1^*, \ldots, p_n^*) \quad \in \quad \boldsymbol{\Delta} := \bigtimes_{i \in N} \Delta_i$$

in which no player has an incentive to deviate, assuming the mixed strategies of all other players remain unchanged.

Definition

A mixed strategy profile $\mathbf{p}^* \in \boldsymbol{\Delta}$ is a Nash equilibrium if for every player $i \in N$

$$U_i(\mathbf{p}^*) \geq U_i(p, \mathbf{p}^*_{-i}) \qquad \text{for all } p \in \Delta_i.$$

NASH EQUILIBRIUM IN MIXED STRATEGIES, EQUIVALENTLY

Which strategy will player i adopt as a reply to the opponents' strategies?

- Best response mapping

$$\mathrm{BR}_i(\mathbf{p}_{-i}) = \operatorname*{argmax}_{p \in \Delta_i} U_i(p, \mathbf{p}_{-i}) \qquad \text{for all } \mathbf{p}_{-i} \in \boldsymbol{\Delta}_{-i}$$

- Best response to $\mathbf{p}_{-i}$ is any strategy $p \in \mathrm{BR}_i(\mathbf{p}_{-i})$

Proposition

The following are equivalent for a mixed strategy profile $\mathbf{p}^* \in \boldsymbol{\Delta}$.

1. $\mathbf{p}^*$ is a Nash equilibrium.
2. $p_i^* \in \mathrm{BR}_i(\mathbf{p}_{-i}^*)$, for each $i \in N$.

NASH'S THEOREM

Theorem
Any finite strategic game has a Nash equilibrium in mixed strategies.

- This is an existential theorem, meaning it proves that a Nash equilibrium exists but does not provide any method for finding one
- Computing a single Nash equilibrium is a very hard problem

HOW TO COMPUTE A NASH EQUILIBRIUM?

1. The case of 2×2 games
2. Support enumeration method for 2 players
3. Multilinear reformulation

INDIFFERENCE PRINCIPLE

Support of a mixed strategy p_i of player i is the set

$$\operatorname{spt} p_i := \{s \in S_i \mid p_i(s) > 0\}.$$

Proposition

Let $\mathbf{p}^*$ be a Nash equilibrium and $i \in N$ be any player.

1. *For every $s, t \in \operatorname{spt} p_i^*$,*

$$U_i(s, \mathbf{p}^*_{-i}) = U_i(t, \mathbf{p}^*_{-i}).$$

2. *For every $s \in \operatorname{spt} p_i^*$,*

$$U_i(s, \mathbf{p}^*_{-i}) = U_i(\mathbf{p}^*).$$

SOLVING 2×2 STRATEGIC GAMES BY INDIFFERENCE PRINCIPLE

Assume that the game has no pure Nash equilibria (2 cases):

- $e < a$ and $b < d$ and $c < g$ and $h < f$
- $a < e$ and $f < h$ and $g < c$ and $d < b$

$$\begin{array}{c|c|c|} & L & R \\ U & a, b & c, d \\ D & e, f & g, h \end{array}$$

Proposition

If the game above has no pure NE, then there exists a completely mixed Nash equilibrium (p_1^*, p_2^*) such that

$$p_1^*(U) = \frac{h - f}{h - f + b - d} \qquad \textit{and} \qquad p_2^*(L) = \frac{g - c}{g - c + a - e}.$$

EXAMPLE: GAMES WITH UNIQUE MIXED NASH EQUILIBRIA

Roommates sharing chores

Alex can cook or clean. Bob shops or takes trash out.

	s	t
c	$2,0$	$0,1$
l	$0,2$	$1,0$

$$p_1^*(c) = \frac{0-2}{(0-2)+(0-1)} = \frac{2}{3}$$
$$p_2^*(s) = \frac{1-0}{(1-0)+(2-0)} = \frac{1}{3}$$

$$U_1(\mathbf{p}^*) = 2 \cdot \frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{3}$$

$$U_2(\mathbf{p}^*) = 2 \cdot \frac{1}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{2}{3} = \frac{2}{3}$$

Matching pennies

	h	t
h	1	-1
t	-1	1

$p_1^*(h) = p_2^*(h) = 1/2$

Penalty kicks

	u	n
u	0.58	0.95
n	0.93	0.70

$p_1^*(u) = 0.38$ and $p_2^*(u) = 0.42$

Data statistics: 0.40 and 0.42

EXAMPLE: GOOD SAMARITAN GAME

Analysis

n players, $S_1 = \cdots = S_n = \{a, \bar{a}\}$, and the utility function of every player i is

$$u_i(\mathbf{s}) = \begin{cases} 9 & s_i = a, \\ 10 & s_i = \bar{a},\ s_j = a \quad \exists j \neq i \\ 0 & \mathbf{s} = (\bar{a}, \ldots, \bar{a}) \end{cases}$$

- There are n pure NE of the form $(a, \bar{a}, \ldots, \bar{a}), \ldots, (\bar{a}, \ldots, \bar{a}, a)$
- In the mixed NE, we get $p_i^*(\bar{a}) = \sqrt[n-1]{0.1} \to 1$ for $n \to \infty$ for each i, and $U_i(\mathbf{p}^*) = 9$
- The probability that nobody helps is $\left(\sqrt[n-1]{0.1}\right)^n \to 0.1$ for $n \to \infty$

SUPPORT ENUMERATION

SUPPORTS OF EQUILIBRIUM STRATEGIES

Proposition

The following are equivalent for a strategy profile $\mathbf{p}^* \in \boldsymbol{\Delta}$.

1. $\mathbf{p}^*$ *is a Nash equilibrium.*
2. $\operatorname{spt} p_i^* \subseteq \mathrm{BR}_i(\mathbf{p}_{-i}^*)$, *for each* $i \in N$.

ARE GIVEN SETS T_1 AND T_2 SUPPORTS?

Linear feasibility problem

Find $p_1 \in \Delta_1$ and $p_2 \in \Delta_2$ such that for $i = 1, 2$:

- $p_i(s_i) = 0$ for every $s_i \notin T_i$
- $U_i(s_i, p_{-i}) \leq e_i$ for all $s_i \notin T_i$
- $U_i(s_i, p_{-i}) = e_i$ for all $s_i \in T_i$

1. If the problem is infeasible, then there is no NE with supports T_1, T_2
2. Any feasible solution is a NE with supports included in T_1, T_2

SUPPORT ENUMERATION METHOD

1. Generate sets $T_1 \subseteq S_1$ and $T_2 \subseteq S_2$
2. Solve the linear feasibility problem for T_1 and T_2
   a. If feasible, then end
   b. If infesible, then 1.

Properties

- The method terminates and finds a single NE
- Order of the generation is crucial for finding NE with small supports

MULTILINEAR REFORMULATION

TEST OF NASH EQUILIBRIUM USING PURE STRATEGIES

Proposition

The following are equivalent for a strategy profile $\mathbf{p}^* \in \Delta$.

1. $\mathbf{p}^*$ *is a Nash equilibrium.*
2. $U_i(\mathbf{p}^*) \geq U_i(s, \mathbf{p}^*_{-i})$ *for all* $s \in S_i$.

OPTIMIZATION REFORMULATION

- Vector variable p_i represents a mixed strategy of player $i \in N$
- Real variable e_i represents an equilibrium value for player $i \in N$

Multilinear program

Minimize

$$\sum_{i \in N} (e_i - U_i(\mathbf{p}))$$

subject to

- $e_i - U_i(s, \mathbf{p}_{-i}) \geq 0$ for each $i \in N$ and every $s \in S_i$
- $p_i \in \Delta_i$ for each $i \in N$
- $e_i \in \mathbb{R}$ for each $i \in N$

PROPERTIES OF THE MULTILINEAR PROGRAM

The following are equivalent:

1. $\mathbf{p}^*$ is a Nash equilibrium with $e_i = U_i(\mathbf{p}^*)$
2. $\mathbf{p}^*$ is a minimizer of the multilinear program with optimal value 0

The key observation (Fischer and Gupte, 2022)

A solver may quickly find a feasible solution with an objective value of 0, but takes considerable time to confirm that no feasible solution has an objective value < 0.

FROM MULTILINEAR PROGRAM TO FEASIBILITY PROBLEM

- Add the constraint expressing non-positivity of the objective
- Any feasible solution to the problem below is a NE

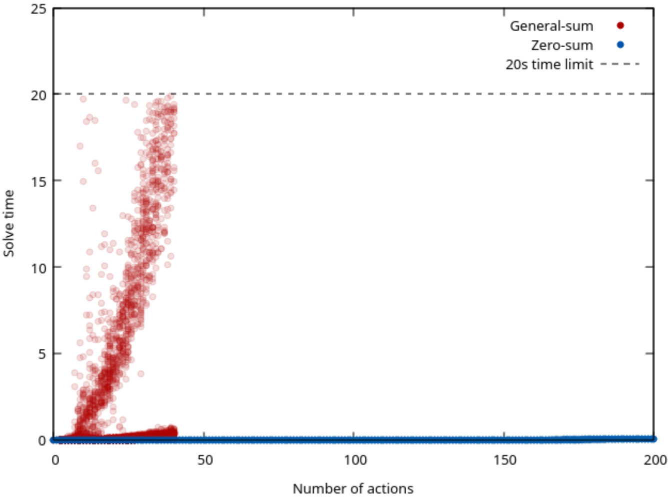
Multilinear feasibility problem

Find $p_1, \ldots, p_n$ satisfying

- $\sum_{i \in N} (e_i - U_i(\mathbf{p}^*)) \le 0$
- $e_i - U_i(s, \mathbf{p}_{-i}) \ge 0$ for each $i \in N$ and every $s \in S_i$
- $p_i \in \Delta_i$ for each $i \in N$
- $e_i \in \mathbb{R}$ for each $i \in N$

MULTILINEAR METHOD VS BASELINE LP FOR ZERO-SUM GAMES

- RCI cluster AMD partition limited to 4 threads and 20s
- Julia + JuMP + Gurobi, 300 randomly generated games for each $n \leq 40$



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