Extensive-form games

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Game theory achievements

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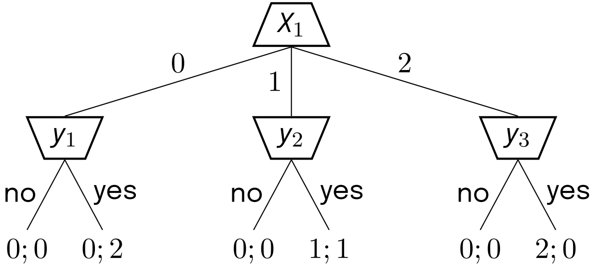
- Deep Blue – Chess (1997)
- AlphaGo – Go (2017)
- Deepstack and Libratus – Poker (2017)
- OpenAI Five – DotA II (2019)
- AlphaStar – Starcraft II (2019)
- DeepNash – Stratego (2022)
- Cicero – Diplomacy (2022)

Multi-agent Sequential Decision-making

- Making decisions sequentially based on current information is more often associated with games.
- Pure strategy in sequential game needs to reflect all possible situations we can encounter in game.
- Pure strategy has to assign single action to each situation that can happen.
- Exponential growth of pure strategies based on the size of the game.

Extensive-form game

Using a tree structure seems more natural for sequential problems. This representation is called extensive-form game.



Extensive-form game

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Extensive-from game (EFG) is defined by:

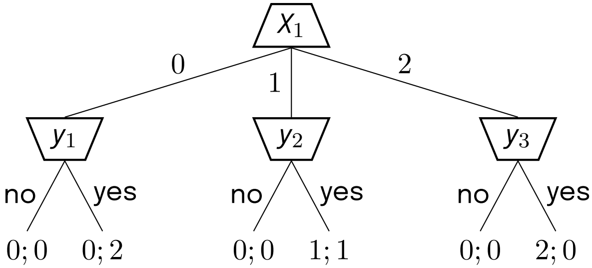
- Player set $\mathcal{N} = \{1, \dots, n\}$
- Actions $\mathcal{A} = \bigcup_{i \in \mathcal{N}} \mathcal{A}_i$, where $\mathcal{A}_i$ is action set of player i
- Decision nodes (histories) $\mathcal{H}$
- Terminal nodes $\mathcal{Z}$
- Player function $N : \mathcal{H} \to \mathcal{N}$
- Action function $A : \mathcal{H} \to 2^{\mathcal{A}}$
- Transition function $\mathcal{T} : \mathcal{H} \times \mathcal{A} \to \mathcal{H} \cup \mathcal{Z}$
- Utility function $u : \mathcal{Z} \to \mathbb{R}^{|\mathcal{N}|}$

A pure strategy of player i in EFG is assignment of single action for each decision node, in which player i acts

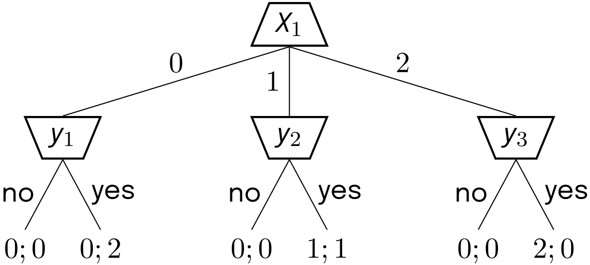
$$\mathcal{S}_i := \bigtimes_{h \in \mathcal{H}, N(h) = i} A(h)$$

Example

What are the actions and pure strategies in this game?



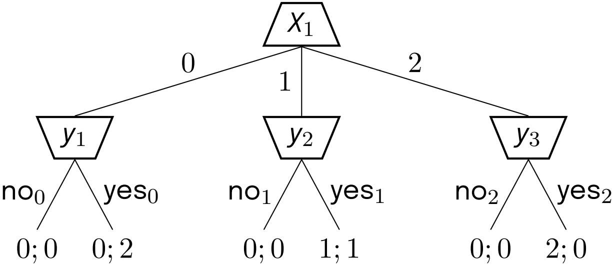
Example



$\mathcal{A}_1 = \{0, 1, 2\}, \mathcal{S}_1 = \{(0), (1), (2)\}$
$\mathcal{A}_2 = \{\text{no}, \text{yes}\}, \mathcal{S}_2 = \{(\text{no}, \text{no}, \text{no}), (\text{no}, \text{no}, \text{yes}), \ldots (\text{yes}, \text{yes}, \text{yes})\}, |\mathcal{S}_2| = 8$
Note that in each decision node y_1, y_2, y_3 player is doing a different decision.

Labelling actions

We often use different labels to uniquely identify actions resulting from different histories

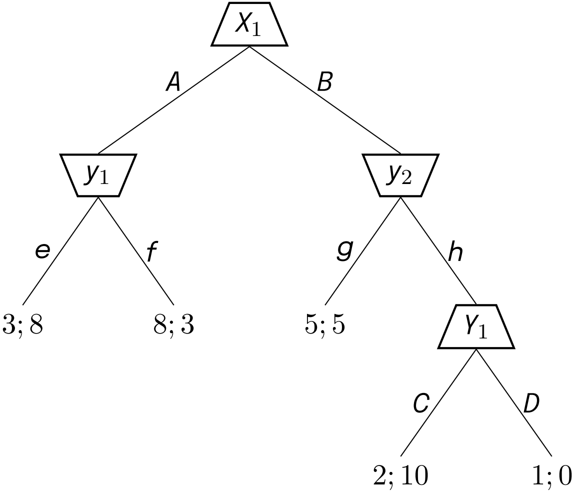


$\mathcal{A}_2 = \{\mathsf{no}_0, \mathsf{yes}_0, \mathsf{no}_1, \mathsf{yes}_1, \mathsf{no}_2, \mathsf{yes}_2\},$
$\mathcal{S}_2 = \{(\mathsf{no}_0, \mathsf{no}_1, \mathsf{no}_2), (\mathsf{no}_0, \mathsf{no}_1, \mathsf{yes}_2), \ldots (\mathsf{yes}_1, \mathsf{yes}_2, \mathsf{yes}_3)\}$

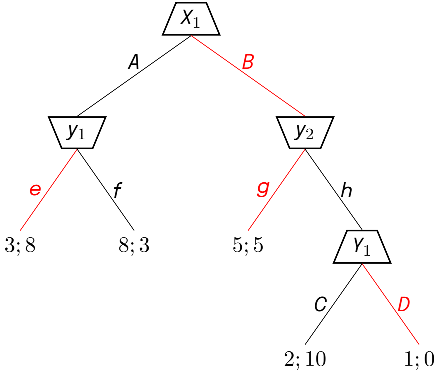
Converting EFG to NFG

- Generate all pure strategies for both players
- Compute utilities corresponding to each pair of those strategies
- Create utility function based on those computed utilities
- Nash equilibrium in this underlying normal-form game is Nash equilibrium in the extensive-form game

Converting EFG to NFG



Converting EFG to NFG



	e, g	e, h	f, g	f, h
A, C	$3, 8$	$\mathbf{3, 8}$	$8, 3$	$8, 3$
A, D	$3, 8$	$\mathbf{3, 8}$	$8, 3$	$8, 3$
B, C	$5, 5$	$2, 10$	$5, 5$	$2, 10$
B, D	$\mathbf{5, 5}$	$1, 0$	$5, 5$	$1, 0$

Rationality of Nash equilibria in EFG

- Some Nash equillibria in Extensive-form games do not have to be rational in all parts of the game tree independently.
- Player may choose irrational actions in parts of the tree, that are outside of the parts, where the Nash equilibrium plays.
- We can use some refinement of the Nash equilibrium, that ensures this rationality in all decision points.
- In EFGs with perfect information this refinement is called **Subgame perfect equilibrium**.
- It can be found algorithmically, by traversing the game tree from bottom and always choosing the action that yields the highest expected utility to each player.
- This algorithms is called Backward Induction, but in two-player zero-sum games with perfect information it is known as minimax.

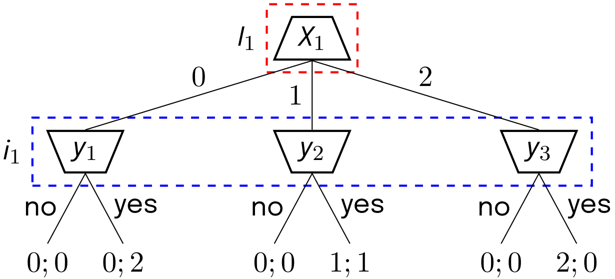
Imperfect Information EFGs

- Player set $\mathcal{N} = \{1, \dots, n\} \cup c$
- Actions $\mathcal{A} = \bigcup_{i \in \mathcal{N}} \mathcal{A}_i$, where $\mathcal{A}_i$ is action set of player i
- Decision nodes (histories) $\mathcal{H}$
- Terminal nodes $\mathcal{Z}$
- Player function $N : \mathcal{H} \to \mathcal{N}$
- Information sets $\mathcal{I} = (\mathcal{I}_1, \dots, \mathcal{I}_n)$, h, h' belong to the same infoset $I_i \in \mathcal{I}_i$ of player i, if it cannot distinguish between them
- Action function $A : \mathcal{H} \to 2^{\mathcal{A}}$, Since player i cannot distinguish between histories in a same infoset I_i, it reqiures same available actions in each of those histories. We often use $A(I_i) := A(h)$
- Transition function $\mathcal{T} : \mathcal{H} \times \mathcal{A} \to \mathcal{H} \cup \mathcal{Z}$
- Utility function $u : \mathcal{Z} \to \mathbb{R}^{|\mathcal{N}|}$

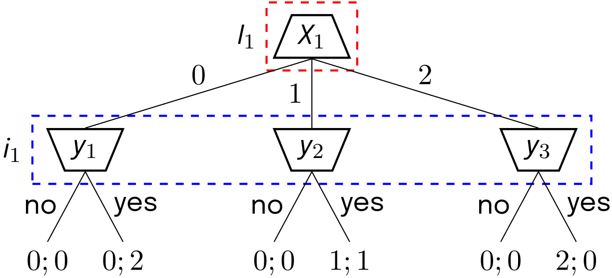
Chance player

- Chance player can be viewed as an another player in a game that has fixed unchangable policy throughout the game, known to all the other players.
- In this case the transition function for chance player is defined analogously as for all the other players.
- Second equivalent way is to define a separate transition function, that is exclusive to the chance player and is publicly known by all the players.
- This does not mean that the outcome of chance is known to all the players.
- Imagine Poker as an example, the probability of dealing a card is known, but all players do not observe which card was dealt.

Example of Imperfect Information EFG



Example of Imperfect Information EFG



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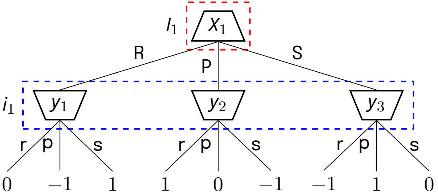
Nash Equilibria in Imperfect Information EFGs



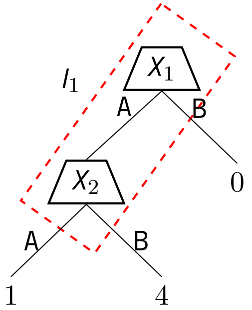
	r	p	s
R	0	-1	1
P	1	0	-1
S	-1	1	0

- Imperfect information games do not have to contain pure Nash equillibria as evidenced by the Rock-Paper-Scissors example.
- Every finite game can be represented as an imperfect information EFG.

Behavioral strategies

- Mixed strategy is a probability distribution between all pure strategies.
- In games it is more natural to think about strategies independently in each decision point.
- These strategies are called Behavioral strategies
- Behavior strategy is a mapping $\pi : \mathcal{I} \rightarrow \Delta A(h)$
- In some games, the behavioral strategy and mixed strategy do not coincide.

Example



- Mixed strategy is a probability distribution on pure strategies.
- Playing mixed strategy corresponds to selecting a single pure strategy at the beginning of the game bsaed on the corresponding probabilities
- Playing mixed strategy $p(A) = p(B) = 0.5$, results in expected value $0.5 \cdot 1 + 0.5 \cdot 0 = 0.5$
- Behavioral strategy gives for each infoset probability distribution across the available actions.
- Playing behavioral strategy corresponds to selecting a single action when facing a decision in some decision node based on the corresponding probabilities
- Playing behavioral strategy $\pi(I_1, A) = \pi(I_1, B) = 0.5$, results in expected value $0.5 \cdot 0 + 0.5 \cdot (0.5 \cdot 1 + 0.5 \cdot 4) = 1.25$.

Perfect Recall

- No player forgets any information throughout the game.
- For any two histories $h, h' \in I_i$, that were formed with trajectories $h_0 a_0 \ldots a_n h$ and $h'_0 a'_0 \ldots a'_m h'$ it has to hold
  - $n = m$
  - for all $0 \leqslant j \leqslant n$, h_j and h'_j are in the same infoset
  - for all $0 \leqslant j \leqslant n$ if $N(h_j) = i$, then $a_j = a'_j$
- This is a standard assumption, required by most of the algorithms that solve imperfect information games

Summary

- Extensive-form is a more natural representation of sequential games.
- In perfect information EFG, there is at least one pure Nash equilibrium.
- Sequentially rational refinement of Nash equilibrium, subgame perfect equilibrium can be found with single traversal of the tree from bottom.
- Every finite game can be represented as an imperfect information EFG
- Imperfect information EFGs do not have to contain pure Nash equilibrium.
- Behavioral strategies in imperfect information EFGs are fundementally different than mixed strategies.
- In games with perfect recall, where neither player forgets any information, behavioral and mixed strategies are the same.