The proof of indifference principle

By contradiction, let there be $s, t \in S_i$ with $U_i(s, p_{-i}^*) > U_i(t, p_{-i}^*)$. Define a new mixed strategy

$$q_i(a) := \begin{cases} 0 & a = t \\ p_i^*(s) + p_i^*(t) & a = s \\ p_i^*(a) & a \in S_i,\ a \neq s, t \end{cases}$$

Expanding $U_i(q_i, p_{-i}^*)$ we can easily verify that

$$U_i(q_i, p_{-i}^*) > U_i(p_i^*, p_{-i}^*)$$

which means that $p^* = (p_i^*, p_{-i}^*)$, is not a Nash equilibrium.

2x2 strategic game

If there is no pure NE, then there must be a mixed NE in which both players use each strategy with positive probability.

Let $p := p_1^*(U)$. Using the indifference principle,

$$U_2(p, L) = U_2(p, R).$$

$$bp + f(1-p) = dp + h(1-p)$$

$$(b - f + h - d)p = h - f$$

$$p = \frac{h - f}{h - f + b - d}$$

Call the police or not?

Pure NE:

1) $(n, \dots, n)$ is not a NE since any $i \in N$ is better off helping

2) If there are at least two players helping, then one of them is better off deviating

3) The only pure NE is

$$s_i = h \quad \text{for some } i \in N$$
$$s_j = n \quad \forall j \neq i.$$

If the game has a mixed NE, then (by symmetry of the game) every player uses the same mixed strategy. We use the indifference principle:

$$U_i(\bar{h}, p_{-i}) = \underbrace{U_i(h, p_{-i})}_{9}$$

$$\parallel$$

$$\sum_{s_{-i} \in S_{-i}} u_i(\bar{h}, s_{-i}) \prod_{j \neq i} p_j(s_j) =$$

$$10\, P(\{s_{-i} \in S_{-i} \mid \exists j \neq i : s_j = h\}) =$$

$$10\,(1 - P(\{s_{-i} \in S_{-i} \mid s_j = \bar{h}\ \forall j \neq i\})) =$$

$$10\,(1 - p^{n-1}), \text{ where } p := p_i(\bar{h}).$$

We solve this equation for p and obtain $p = \sqrt[n-1]{0.1}$.

The value of this NE is

$$U_i(p, \dots, p) = 9\left(1 - \sqrt[n-1]{0.1}\right) + 10\left(\sqrt[n-1]{0.1}\left(1 - \left(\sqrt[n-1]{0.1}\right)^{n-1}\right)\right) = \underline{\underline{9}}$$

Finally, we need to verify that the mixed strategy profile $(p, \dots, p)$ is a NE. But this is easy to see since

$$U_i(p, \dots, p) = 9 = U_i(h, p_{-i}) = U_i(\bar{h}, p_{-i})$$

Supports of equilibrium strategies

1. ⇒ 2.

Let $s \in S_i$ satisfy $p_i^*(s) > 0$. Then

$$U_i(s, p_{-i}^*) = U_i(p^*) = \max_{s' \in S_i} U_i(s', p_{-i}^*),$$

where follows from the indifference principle and is a consequence of the definition of NE together with the fact that function $U_i(\cdot, p_{-i}^*)$ is linear over the standard simplex.

2. ⇒ 1. This is proved analogously to the proof of indifference principle.

NE using pure strategies

1. ⇒ 2. Since any pure strategy is mixed, this is a consequence of the definition.

2. ⇒ 1. Any mixed strategy is a unique convex combination of pure strategies. Since such combinations preserve inequalities 2., we obtain the inequality defining NE.